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College of Engineering
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EE227B — Convex Optimization
Lecture Notes

Calvin Chi
Rodrigo Henriquez-Auba

Contents

1 Introduction

The complexity of an optimization problem is measured by the number of basic operations, which depends on the number of decision variables to solve for n, number of constraints m, and miscellaneous costs such as cost of evaluating derivatives z.

The difficulty of an optimization problem is measured by its complexity. Let n be the number of basic operations to solve a problem, then $O(n)$ would be an easy problem and exponential growth such as $O(2^n)$ would be a hard problem. Below is an example $O(2^n)$ problem

$$\begin{aligned} \arg\min_x \quad & c^T x \\ & x_i^2 = 1, \quad i = 1, \dots, n \end{aligned}$$

because there are 2^n possible solutions.

The main classes of optimization problems include linear programs (LP), quadratic programs (QP), quadratically constrained quadratic programming (QCQP), second-order cone programming (SOCP), semidefinite programming (SDP), and conic programs. They relate to each other by

$$\text{LP} \subset \text{QP} \subset \text{QCQP} \subset \text{SOCP} \subset \text{SDP} \subset \text{conic programs}$$

Solving optimization problems often involve reformulating hard problems into easier problems.

2 Convexity

Given a set of vectors $x_1, \dots, x_n$ in $\mathbb{R}^n$, the combination $\sum_i \alpha_i x_i$ is

- Affine if $\sum_i \alpha_i = 1$.
- Convex if $\sum_i \alpha_i = 1$ and $\alpha_i \geq 0$ for all i.
- Conical if $\alpha_i \geq 0$ for all i.

A set of vectors is affine, convex, or conical if any affine, convex, or conical combination of n vectors is also in the same set.

2.1 Example Convex Sets

- **Half-spaces**. Let $a^T x = b$ define a hyperplane, then $a^T x \geq b$ or $a^T x \leq b$ is the corresponding half-space.
- **Polyhedron**. A polyhedron can be described as the set

$$P = \{x \mid Ax \preceq b, Cx = d\}$$

where $A \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{p\times n}$.

- **Norm balls**. A norm ball can be described as the set

$$B(x_c, r) = \{x \mid ||x - x_c|| \leq r\}$$

  To prove this, use the triangle inequality and positive homogeneity of norms.

- **Ellipsoids**. An ellipsoid can be described as the set

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$$

  where $P \in \mathbb{S}^n_{++}$[1].

2.2 Set Operations Preserving Convexity

Let $f : \mathbb{R}^n \to \mathbb{R}^m$, then dom f is the set of values x where $f(x)$ is defined. The range of f is the set of all values $f(x)$ where $x \in \text{dom} f$. The following are operations on convex sets that preserve convexity.

- **Intersection**. Intersection of convex sets are also convex. However, the union of convex sets is generally not convex.
- **Affine transformation**. Let $f(x) = Ax + b$ be a function where $f : \mathbb{R}^n \to \mathbb{R}^m$. Let S denote a convex set, then the image of S ($f(s) = \{f(x)|x \in S\}$) under f is also a convex set. The inverse image of S under f is also convex ($f^{-1}(s) = \{x|f(x) \in S\}$).
- **Projection**: the projection of members of a convex set to a lower dimensional space results in another convex set.
- **Linear fractional transformation**. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be of the form

$$f(x) = \frac{Ax + b}{c^T x + d}$$

  For $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $d \in \mathbb{R}$. dom $f = \{x|c^T x + d > 0\}$. Then if S is a convex set, then the image $f(S)$ is also a convex set. The inverse image $f^{-1}(S)$ is also convex.

2.3 Convex Functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\forall x, y$ and $\forall \alpha \in [0, 1]$,

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \tag{1}$$

[1] Here $\mathbb{S}_{++}$ denotes the set of positive definite matrices and $\mathbb{S}_{+}$ denotes the set of positive semidefinite matrices

where f has a convex domain[2]. Strict convexity is achieved if the inequality of Equation 1 is strict. The geometric interpretation of a convex function is shown in Figure 1

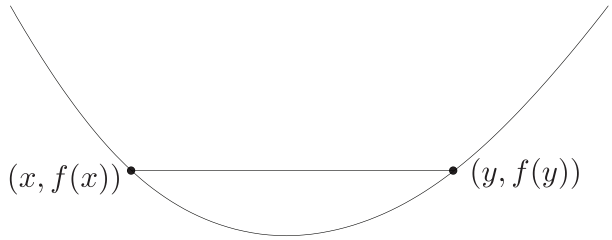

Figure 1: The line segment $\alpha f(x) + (1-\alpha) f(y)$ is above $f(\alpha x + (1-\alpha)y)$.

If the function f is furthermore continuous, then the midpoint theorem states that checking Equation 1 is true for $\alpha = \frac{1}{2}$ is sufficient to establishing convexity of f.

First Order Condition for Convexity. Suppose a function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and continuous, then f is convex if and only if dom f is a convex set and for all $x, y \in \text{dom} f$,

$$f(y) \geq f(x) + \nabla_x f(x)^T (y - x) \tag{2}$$

The geometric interpretation is shown in Figure 2.

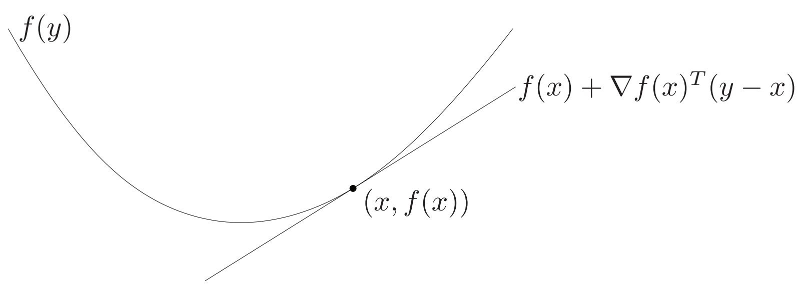

Figure 2: Illustration of the condition that $f(y) \geq f(x) + \nabla_x f(x)^T (y - x)$. Source: Boyd.

Second Order Condition for Convexity. Suppose a function $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable and continuous.Then f is convex if and only if dom f is convex and its Hessian is positive semidefinite (PSD).

$$\nabla_x^2 f(x) \succeq 0 \tag{3}$$

However, a positive definite Hessian is a sufficient but not necessary condition for strict convexity. For instance $f(x) = x^4$ is strictly convex but $\nabla_x^2 f(x) = 12x^2 = 0$ at $x = 0$.

[2]The requirement that the domain of f be a convex set is just to ensure that $f(\alpha x + (1-\alpha)y)$ is defined.

Below are some examples of convex functions that can be verified using the above conditions for convexity.

- **Exponential**. A function of the form $f(x) = e^{ax}$ is convex for $a \in \mathbb{R}$ and strictly convex if $a \neq 0$. This is easily checked with the second order condition for convexity, where $\nabla f^2(x) = a^2 e^{ax} \geq 0$.

- **Powers**.

  1. x^a is convex on $\mathbb{R}_{++}$ if $a \geq 1$ or $a \leq 0$[3].
  2. $-x^a$ is convex on $\mathbb{R}_+$ if $0 \leq a \leq 1$.

- **Logs**. The negative log determinant $-\log \det X$ is convex on domain of PSD matrices. This is a generalization of the statement that $-\log x$ is convex on $\mathbb{R}_{++}$. The negative entropy $x \log x$ is also convex on $\mathbb{R}_{++}$.

Below are additional conditions for establishing convexity.

- Pointwise maximum of a set of convex functions is convex. If $f(x, y)$ is a convex function of x, then for every y in the index set $\mathcal{D}$, $g(x) = \sup_{y \in \mathcal{D}} f(x, y)$ is a convex function in x. Figure 3 illustrates this concept.

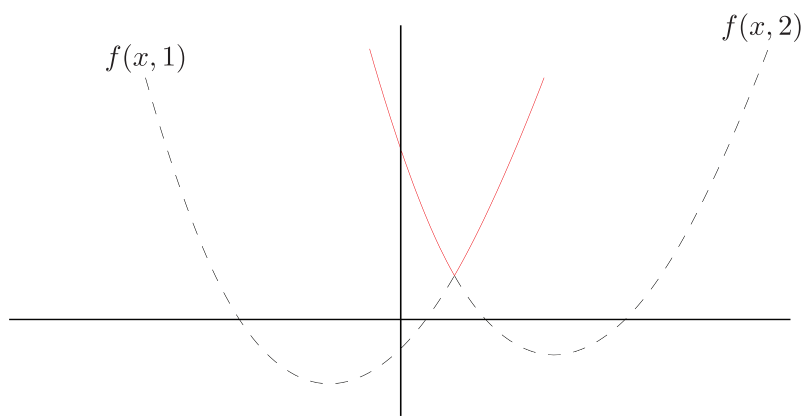

Figure 3: Illustration of the condition that the pointwise maximum of a set of convex functions is convex. Here the index set $\mathcal{D} = \{1, 2\}$ and the red portion of the graph is $g(x) = \sup_{y \in \mathcal{D}} f(x, y)$.

- If $f(x)$ is convex, then $g(x) = f(Ax + b)$ is also convex for arbitrary $A \in \mathbb{R}^{m \times n}$ and $b \times \mathbb{R}^m$.

- Assume $f(x, y)$ is jointly convex in x and y and $\mathcal{D}$ is a convex set. Then $g(x) = \inf_{y \in \mathcal{D}} f(x, y)$ is convex if $g(x)$ is always greater than $-\infty$. To prove this, let $u, v \in \mathrm{dom} f$. By definition of infimum, since $g(u) \leq g(u) + \frac{\epsilon}{2}$ for any $\epsilon > 0$, there exists $y_1 \in \mathcal{D}$ such that $f(u, y_1) \leq g(u) + \frac{\epsilon}{2}$. By the same argument there exists $y_2 \in \mathcal{D}$ such that $f(v, y_2) \leq g(v) + \frac{\epsilon}{2}$. For $\alpha \in [0, 1]$

[3] x^a needs to be defined on $\mathbb{R}_{++}$ because otherwise $x^{-1/2}$ is undefined for $x = 0$.

$$\begin{aligned}
f(\alpha u + (1-\alpha)v) &= \inf_{y\in\mathcal{D}} f(\alpha u + (1-\alpha)v, y) \\
&\leq f(\alpha u + (1-\alpha)v, \alpha y_1 + (1-\alpha)y_2) \\
&\leq \alpha f(u, y_1) + (1-\alpha) f(v, y_2) \qquad \text{since } f(x,y) \text{ is jointly convex in } x, y \\
&\leq \alpha g(u) + (1-\alpha) g(v) + \epsilon
\end{aligned}$$

- Line restriction. A function f is convex if and only if its restriction to any line is convex. That is for every member of the set $\{t|x + ty \in \text{dom} f\}$, $g(t) = f(x + ty)$ is convex. An example of this is that the intersection between a cone and a vertical hyperplane results in another convex function g parameterized by t.
- A function if convex if and only if its epigraph is a convex set.
- A conic combination of convex functions results in another convex function.
- Composition theorem: Let $f : \mathcal{D}_1 \to \mathbb{R}$ and $g : \mathcal{D}_2 \to \mathbb{R}$, where $\mathcal{D}$ denotes domain of a function. Also impose Range$(f) \subseteq \mathcal{D}_2$. Then if f and g are convex and g is non-decreasing, then $g(f(x))$ is also convex. For example, if $f(x)$ is convex, then $e^{f(x)}$ is also convex.
- A quadratic function of the form $f(x) = x^T Ax + b^T x + c$ is convex if A is PSD and strictly convex if A is positive definite. This can be proved with the second-order condition of convexity.

$$\begin{aligned}
f(x) &= x^T Ax + b^T x + c \\
&\Rightarrow \nabla f(x) = 2Ax + b \\
&\Rightarrow \nabla^2 f(x) = 2A \\
&\Rightarrow f \text{ is convex } \Leftrightarrow A \succeq 0
\end{aligned}$$

- Indicator functions. Given a set $\mathcal{D}$ and the indicator function

$$I_{\mathcal{D}}(x) = \begin{cases} 0 & x \in \mathcal{D} \\ \infty & \text{otherwise} \end{cases}$$

  Then $\mathcal{D}$ is a convex set $\leftrightarrow I_D(x)$ is convex. To quickly see this, $\alpha x + (1-\alpha)y \in \mathcal{D}$ for $\alpha \in [0,1]$ and any $x, y \in \mathcal{D}$ by definition that $\mathcal{D}$ is convex. Thus $I_{\mathcal{D}}(\alpha x + (1-\alpha)y) = 0$ always. Given that $\alpha I_{\mathcal{D}}(x) + (1-\alpha) I_{\mathcal{D}}(y) = 0$, the definition of convex functions still holds by strict equality.

Indicator functions are useful in formulating optimization problems. As an example, the following constrained optimization problem in standard form

$$\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & f_0(x) \\
\text{subject to:} \quad & f_i(x) \leq 0, i = 1, \dots, m
\end{aligned}$$

can be expressed as the unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n} \quad f_0(x) + I_{\mathcal{D}}(x)$$

where $\mathcal{D}$ is the feasible set of the constrained optimization problem.

Consider the problem of finding the minimum distance of a point x to a convex set $\mathcal{M}$.

$$g(x) = \inf_{y \in \mathcal{M}} ||x - y||$$

since $||x - y||$ is jointly convex in x, y and $\mathcal{M}$ is convex, by one of the above conditions $g(x)$ is a convex function[4].

Let $||\cdot||$ be a norm on $\mathbb{R}^n$. Its dual norm is

$$||u||_* := \sup_{||x|| \leq 1} u^T x$$

From this definition it can be easily verified that the dual norm of $l2$ norm is the $l2$ norm and the dual norm of $l1$ norm is the ∞ norm[5]. The dual norm is a norm, therefore the dual norm is convex.

3 Conjugates and Subdifferentials

3.1 Conjugates

A Fenchel conjugate of a function $f(x)$ is defined as

$$f^*(z) = \sup_{x \in \text{dom} f} x^T z - f(x) \tag{4}$$

The Fenchel conjugate $f^*(z)$ is convex because the pointwise supremum of a set of affine functions in z is convex. Notice that $f^*(z)$ is convex even if $f(x)$ is not.

Fenchel's inequality states that

$$f(x) + f^*(y) \geq x^T y \tag{5}$$

for all x, y.

Example 1
Let $f(x) = ax + b$ be $f : \mathbb{R} \to \mathbb{R}$. Then

$$f^*(z) = \sup_x zx - (ax + b) = \sup_x (z - a)x - b = \begin{cases} -b & \text{if } z = a \\ \infty & \text{otherwise} \end{cases}$$

Thus $f^*(z) = -b$ with $\text{dom} f^* = \{a\}$.

[4] To see why $||x - y||$ is jointly convex in x, y, for $\alpha \in [0, 1]$, $||[\alpha x_1 + (1-\alpha)x_2] - [\alpha y_1 + (1-\alpha)y_2]|| = ||\alpha(x_1 - y_1) + (1-\alpha)(y_2 - x_2)|| \leq \alpha||x_1 - y_1|| + (1-\alpha)||x_2 - y_2||$. Assume $||\cdot||$ excludes the $l0$ norm.

[5] For dual norm of $l2$ norm, $x^* = u/||u||_2$. For dual norm of $l1$ norm, let $j = \arg\max_i\{u_i, -u_i\}$, then $x^* = \text{sign}(u_j)$

Example 2
If $f(x) = ||x||$, then

$$f^*(z) = I_{||\cdot||_* \leq 1}(z) = \begin{cases} 0 & ||z||_* \leq 1 \\ \infty & ||z||_* > 1 \end{cases}$$

Recall dual norm definition $||z||_* = \sup_{||x|| \leq 1} x^T z$.

- **Case 1**. If $||z||_* \leq 1$, then for any x
$$\begin{aligned} z^T \left(\frac{x}{||x||} \right) &\leq ||z||_* \leq 1 \\ &\Rightarrow z^T x - ||x|| \leq 0 \\ &\Rightarrow f^*(z) = \sup_x z^T x - ||x|| \leq 0 \end{aligned}$$
  which has solution $x^* = 0$ with value $f^*(z) = 0$.
- **Case 2**. If $||z||_* > 1$, then there exists x such that $||x|| \leq 1$ and $x^T z > 1$. Then $f^*(z) = \sup_x \left(z^T x - ||x|| \right) \geq z^T(tx) - ||tx|| = t(z^T x - ||x||)$ for $t \in \mathbb{R}$.
  Thus if $t \to \infty$, then the lower bound $t(z^T x - ||x||) \to \infty$ since $z^T x - ||x|| > 0$. This implies that for $f^*(z) = \sup_x \left(z^T x - ||x|| \right)$, x can be chosen such that $f^*(z) = \infty$.

Exercise 1
Let $f(x) = \frac{1}{2} x^T A x + b^T x + c$ with $A \succ 0$. Show that $f^*(z) = \frac{1}{2}(z - b)^T A^{-1}(z - b) - c$.

Solution
By definition we have

$$f^*(z) = \sup_x \left[x^T y - \left(\frac{1}{2} x^T A x + b^T x + c \right) \right]$$

Let $g(x) := x^T y - \left(\frac{1}{2} x^T A x + b^T x + c \right)$. Since $g(x)$ is concave, it has a global maximum that can be found via the first order optimality condition.

$$\nabla g(x) = 0 \Rightarrow x^* = A^{-1}(z - b)$$

Substitution yields

$$\begin{aligned} f^*(z) &= (x^*)^T z - \frac{1}{2}(x^*)^T A (x^*) - b^T x^* - c \\ &= (z - b)^T A^{-1} z - \frac{1}{2}(z - b)^T A^{-1}(z - b) - b^T A^{-1}(z - b) - c \\ &= \frac{1}{2}(z - b)^T A^{-1}(z - b) - c \end{aligned}$$

where going to the third line from the second line may involve computing the individual terms and canceling each other out.

Exercise 2
Let $f(x) = \sum_{i=1}^n x_i \log x_i$. Show that $f^*(z) = \sum_{i=1}^n e^{z_i - 1}$.

Solution
From definition we have

$$\begin{aligned}
f^*(z) &= \sup_x \left[z^T x - \sum_{i=1}^n x_i \log x_i \right] \\
&= \sum_{i=1}^n \sup_{x_i} \left[z_i x_i - x_i \log x_i \right] \\
&\Rightarrow x_i^* = e^{z_i - 1} && \text{by first order condition: } z_i x_i - x_i \log x_i \text{ is convex} \\
&\Rightarrow f^*(z) = \sum_{i=1}^n e^{z_i - 1} && \text{By substitution with } x_i^*
\end{aligned}$$

Exercise 3
Let $f(X) = -\log\det(X)$, where dom $f = \mathbb{S}_{++}^n$. Show that $f^*(Z) = -\log(\det(-Z)) - n$, where Z belongs to a set of negative definite matrices.

Solution
Let $g(Z) = \text{trace}(XZ) + \log\det(X)$. From definition we have

$$\begin{aligned}
f^*(Z) &= \sup_X g(Z) \\
&\Rightarrow \nabla_X g(Z) = Z + X^{-1} = 0 && -\log\det(X) \text{ is convex} \\
&\Rightarrow X^* = (-Z)^{-1} \\
&\Rightarrow f^*(Z) = \text{trace}(-(A^{-1}A)) + \log\det((-Z)^{-1}) && \text{By substitution with } X^* \\
&\Rightarrow f^*(Z) = -n + \log[\det(-Z)]^{-1} \\
&\Rightarrow f^*(Z) = -n - \log[\det(-Z)]
\end{aligned}$$

Where in the second line, we used the matrix derivative rules that $\nabla_A \text{trace}(AB) = B^T$ and $\nabla_A \det(A) = \det(A)(A^{-1})^T$. The fourth line uses the fact that $(cA)^{-1} = c^{-1}A^{-1}$ for constant c. The fifth line uses the fact that $\det(A^{-1}) = 1/\det(A)$.

3.2 Subdifferentials

A vector $g \in \mathbb{R}^n$ is a subgradient at point y if for all $x \in \text{dom} f$, it holds that

$$f(x) \geq f(y) + g^T(x - y) \tag{6}$$

The subgradient is a generalization of the concept of a gradient for a convex function. If the convex function is differentiable, then the subgradient is unique. However, if the convex function is not differentiable everywhere, then there exists an infinite number of subgradients $g_1, \ldots, g_\infty$ at point y. Figure 4 illustrates an example convex function that is not differentiable everyone.

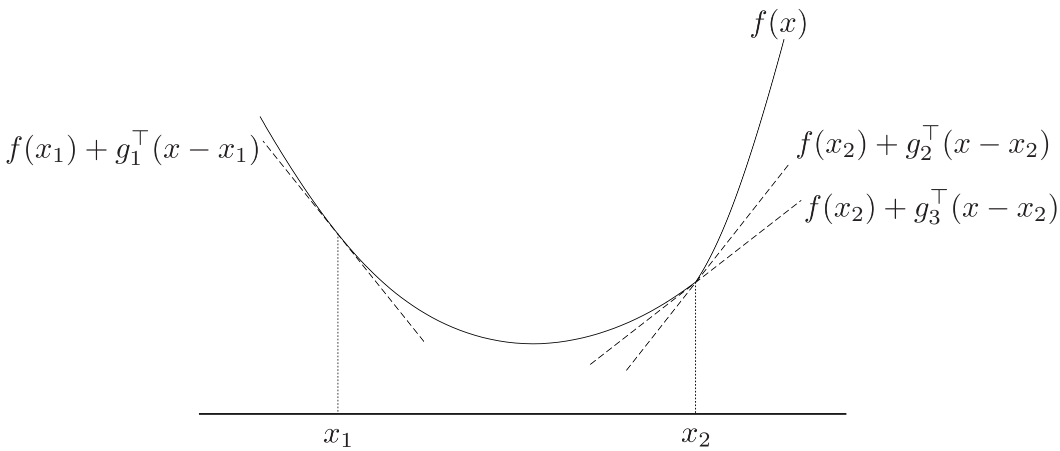

Figure 4: Example convex function that is not differentiable everywhere. There are multiple subgradients at x_2.

A subgradient is a global linear under-estimator of a function f at $(y, f(y))$ because of the inequality in Equation 6. In general local under-estimators are also global under-estimators if f is convex.

A subdifferential is defined as the set of all subgradients at y for function f and is denoted $\partial f(x)$.

Theorem 3.2.1 A subdifferential $\partial f(y)$ is a closed convex set and possibly empty.

Proof. From definition $f(x) - f(y) \geq (x - y)^T g$, where y is fixed. For each x, the set of g's is a half-space, which is a convex set. A subdifferential can then be expressed as the intersection of such half-spaces

$$\cap_x \{g | f(x) \geq f(y) + g^T(x - y)\}$$

Since the intersection of convex sets is another convex set, $\partial f(y)$ is a convex set.

Example 1
Let f be a function with dom f on $\mathbb{R}_+$ and defined as

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$$

At $x = 0$, the subdifferential $\partial f(x) = \emptyset$. This is illustrated in Figure 5.

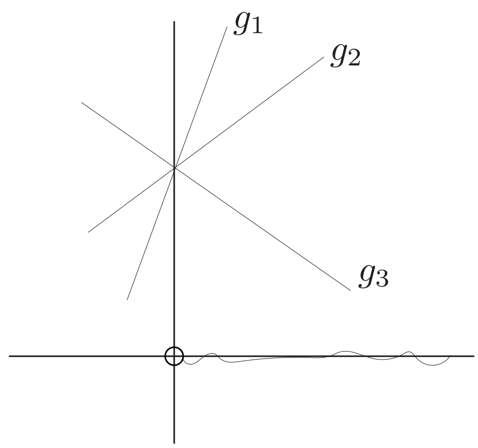

Figure 5: Example function f where $\partial f(x)$ at $x = 0$ is the empty set.

> **Theorem 3.2.2** If $x \in$ interior of dom f, then $\partial f(x)$ at x is nonempty and bounded where f is convex.

Example 2
If $f(x) = |x|$, then the subdifferential is

$$\partial f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ [-1, 1] & x = 0 \end{cases}$$

In all cases, the subdifferential is nonempty and bounded.

> **Theorem 3.2.3** If f is differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$. If $\partial f(x) = \{g\}$, then $f(x)$ is differentiable at x and $g = \nabla f(x)$.

Example 3 Let $f = ||x||_2$ be the Euclidean norm. Find $\partial f(x)$. There are two cases to consider - $x = 0$ and $x \neq 0$.

When $x \neq 0$

$$\frac{\partial f(x)}{\partial x_i} = \frac{x_i}{||x||_2}$$

From this it can be inferred that $\nabla f(x) = x/||x||_2$. When $x = 0$, this corresponds to the case when $y = 0$ in Equation 6, which leads to

$$||x||_2 \geq g^T x$$

We know from dot products that the above inequality is met with equality when $g = \frac{x}{||x||_2}$, and that with any other vector $y \neq x$ with $||y||_2 \leq 1$, the inequality is strict. Thus it can be concluded that

$$\partial f(x) = \begin{cases} ||x||_2^{-1} x & x \neq 0 \\ \{z \mid ||z||_2 \leq 1\} & x = 0 \end{cases}$$

Theorem 3.2.4 If $f(x) = \max_x\{f_1(x), \ldots, f_k(x)\}$, then $\partial f(x)$ is the convex hull of union of $\partial_i f(x)$ for all $i \in \mathcal{L}(x)$, where $\mathcal{L}(x)$ is the set of indices of active functions at x

$$\mathcal{L}(x) = \{i \mid f(x) = f_i(x)\}$$

Consider the $f(x) = \max_x\{f_1(x), f_2(x)\}$ in Figure 6.

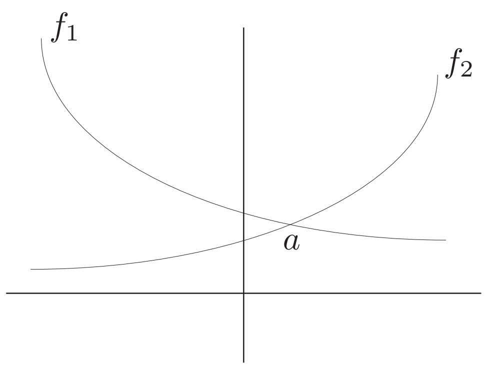

Figure 6: Example functions f_1 and f_2 where both functions are active at $x = a$.

Then

$$\partial f(x) = \begin{cases} \{\nabla f_1(x)\} & x < a \\ \{\nabla f_2(x)\} & x > a \\ [\nabla f_1(x), \nabla f_2(x)]^T \theta & x = a \end{cases}$$

Example 4
Let $f(x) = \max_{1 \leq i \leq k}\{a_i^T x + b_i\}$. Then $\partial f(x) =$ convex hull of $\{a_i | \mathcal{L}(x)\}$ since $\partial f_i(x) = \{a_i\}$.

4 Optimization Problems

An optimization problem of the form

$$
\begin{aligned}
p^\star = \min_{x \in \mathbb{R}^n} \quad & f_0(x) \\
\text{subject to:} \quad & f_i(x) \le 0, i = 1, \dots, m \\
& h_i(x) = 0, i = 1, \dots, p
\end{aligned} \tag{7}
$$

is called a convex optimization problem if

- The objective function f_0 is convex
- The functions f_i, $i = 1, \dots, m$ are convex
- The functions h_i, $i = 1, \dots, p$ are affine

A convex optimization problem can be reformulated as follows

- Conversion between maximization and minimization problems. $\arg\max g(x) = \arg\min -g(x)$.
- An equality constraint can be expressed as two inequality constraints.

  $$g(x) = 0 \Leftrightarrow \begin{cases} g(x) \le 0 \\ -g(x) \le 0 \end{cases}$$

- An inequality constraint can be expressed as an equality constraint. $g(x) \le 0 \Leftrightarrow g(x) + z^2 = 0$ for $z \in \mathbb{R}$.
- Change of variables. Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one, with $\phi(\text{dom}\phi) \supseteq \mathcal{D}$, where $\mathcal{D}$ is the problem domain of the optimization problem in problem 7. We define functions $\tilde{f}_i = f_i \circ \phi$ and $\tilde{h}_j = h_j \circ \phi$ as

  $$\tilde{f}_i(z) = f_i(\phi(z)), \quad i = 0, \dots, m, \qquad \tilde{h}_j(z) = h_j(\phi(z)), \quad j = 1, \dots, p$$

  Then the optimization problem

  $$
  \begin{aligned}
  \min_{z \in \mathbb{R}^n} \quad & \tilde{f}_0(z) \\
  \text{subject to:} \quad & \tilde{f}_i(z) \le 0, i = 1, \dots, m \\
  & \tilde{h}_j(z) = 0, j = 1, \dots, p
  \end{aligned} \tag{8}
  $$

  is equivalent to problem 7. Two problems are equivalent if the solution of one problem can be readily found from solution to another. In change of variables, $z = \phi^{-1}(x)$ solves problem 8 while $x = \phi(z)$ solves problem 7.

- Transformation of objective and constraint functions. Suppose
  - $\psi_0 : \mathbb{R} \to \mathbb{R}$ is strictly increasing.
  - $\psi_i : \mathbb{R} \to \mathbb{R}$; $\psi_i(y) \le 0 \Leftrightarrow y \le 0; i = 1, \dots, m$.
  - $\psi_i : \mathbb{R} \to \mathbb{R}$; $\psi_i(y) = 0 \Leftrightarrow y = 0; i = m + 1, \dots, m + p$.

Then the new problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \psi_0(f_0(x)) \\ \text{subject to:} \quad & \psi_i(f_i(x)) \leq 0, i = 1, \dots, m \\ & \psi_{j+m}(h_j(x)) = 0, j = 1, \dots, p \end{aligned} \tag{9}$$

is equivalent to problem 7 because the feasible sets and optimal points are identical. An example of this is $\arg\min_x ||Ax - b||_2 = \arg\min_x ||Ax - b||_2^2$, where $\psi_0(x) = x^2$ is strictly increasing on $x \in \mathbb{R}_+$, which is the case since $f_0(y) = ||y||_2 \geq 0$.

- Epigraph form. Any optimization problem can be converted to another optimization problem whose objective is linear

$$\begin{aligned} \min_{x\in\mathbb{R}^n, t\in\mathbb{R}} \quad & t \\ \text{subject to:} \quad & f_0(x) - t \leq 0 \\ & f_i(x) \leq 0, i = 1, \dots, m \\ & h_j(x) = 0, j = 1, \dots, p \end{aligned} \tag{10}$$

at optimality, $t^* = f_0(x^*)$. The problem does not set $f_0(x) = t$ because an optimization algorithm needs to guess $x \in \mathbb{R}^n, t \in \mathbb{R}$ such that $f_0(x) \leq t$.

Given $f : \mathbb{R}^n \to \mathbb{R}$ with domain $\mathcal{X}$, a stationary point[6] can be categorized into the following types of points.

- Local minimum: a point $x^* \in \mathbb{R}^n$ such that there exists $R > 0$ where $f(x^*) \leq f(x)$ for all $x \in \mathcal{X} \cap \{y \mid ||y - x^*||_2 \leq R\}$.
- Global minimum: a point $x^* \in \mathbb{R}^n$ that is feasible and gives lowest value possible for $f(x)$.

A few terms need to be defined for $x \in \mathcal{X}$ for an optimization problem of the general form in problem 7.

- **Feasible Solution**: satisfies all constraints and belongs to domain of optimization problem, where domain = $\text{dom } f_0(x) \cap \text{dom } f_1 \cap \cdots \cap \text{dom } f_m \cap \text{dom } h_1 \cap \cdots \cap \text{dom } h_p$.
- **Optimal Solution**: infimum of $f_0(x)$ over all feasible solutions, with optimal values $\in \{-\infty, \text{finite real value}, \infty\}$. An optimal value of $-\infty$ is described as unbounded below and an optimal value of ∞ is described as infeasible. Example optimization problems with $-\infty$ optimal values include $\min_{x\in\mathbb{R}_{++}} \log x$ and $\min_{x\in\mathbb{R}} x$.

4.1 Convex Optimization

For a convex optimization, satisfying the second order condition is necessary and sufficient for guaranteeing a global minimum. However, the second order condition is necessary and sufficient only for guaranteeing a local minimum.

[6]A stationary point for a function f is a point $x^* \in \mathbb{R}^n$ where $\nabla_x f(x^*) = 0$.

A convex optimization problem involves minimization of a convex function over a convex set. According to problem 7, this means that $f_0, f_1, \ldots, f_m$ are convex functions and $h_1, \ldots, h_p$ are affine functions. The feasible set of a convex optimization is convex because it is the intersection of sublevel sets and hyperplanes, both of which are convex sets. The set satisfying $f_i \leq 0$ for all i is a convex sublevel set because f_i for all i are convex.

Example 1
The optimization problem

$$\begin{aligned} \min_x \quad & x_1^2 + x_2^2 \\ \text{subject to:} \quad & x_1^3 \leq 0 \\ & x_1 + x_2 \geq 1 \end{aligned}$$

is not convex because $f(x) = x^3$ is not convex on $\mathbb{R}_-$. However, on observing that $x^3 \leq 0$ if and only if $x \leq 0$ allows us to replace the constraint $x_1^3 \leq 0$ with $x_1 \leq 0$, which turns the optimization problem into a convex optimization problem.

Theorem 4.1.1 Every local minimum of a convex optimization problem is a global minimum.

Proof. Proof by contradiction. Suppose x is a locally optimal point and y is an optimally global point with $f(y) < f(x)$. By definition of local optimality, $\exists\, R > 0$ such that $f(x) < f(z)$ for all $||x - z||_2 \leq R$. A proof by contradiction would involve showing it is possible to choose z such that $||x - z||_2 \leq R$ is satisfied for any $R > 0$, but $f(z) < f(x)$.

$$z = \theta y + (1 - \theta)x \quad \text{with} \quad \theta = \frac{R}{2||x - y||_2}$$

Then

$$\begin{aligned} ||x - z||_2 &= \left|\left| x - \left(\frac{R}{2||x - y||_2} y + \left(1 - \frac{R}{2||x - y||_2}\right) x \right) \right|\right|_2 \\ &= \left|\left| \frac{R}{2||x - y||_2}(x - y) \right|\right|_2 \\ &= R/2 \leq R \end{aligned}$$

So for any given $R > 0$, $\theta \in [0, 1]$ can be chosen such that z is less than R in Euclidean distance to x. By convexity of f we have

$$f(z) = f(\theta y + (1 - \theta)x) \leq \theta f(y) + (1 - \theta) f(x) < f(x)$$

where the last strict inequality is true because $f(y)$ is the global minimum. However, $f(z) < f(x)$ is a contradiction to assuming x is a local minimum, thus such a local minimum cannot exist for convex objective function f. Intuitively, this proof states that any point on the line segment between x and y has objective value less than $f(x)$. To conclude the proof, since local minima cannot exist for convex objective functions, only global minima can exist for convex objective functions.

Below are a few additional conditions for global minimum of a convex function f.

Theorem 4.1.2 Consider a point x that is in the interior of the feasible set $\mathcal{X}$.

$$x \text{ is a global minimum} \Leftrightarrow \nabla f_0(x) = 0$$

Theorem 4.1.3 Consider a point x in the feasible set $\mathcal{X}$.

$$x \text{ is a global minimum} \Leftrightarrow (\nabla f_0(x))^T(y-x) \geq 0 \quad \forall y \in \mathcal{X}$$

for convex function f_0.

Proof.

- Forward direction: contrapositive proof. Suppose $(\nabla f_0(x))^T(y-x) < 0$. Consider the point $z(t) = ty + (1-t)x$ with $t \in [0,1]$. Notice that

$$\frac{d}{dt} f_0(z(t)) = \nabla f_0(x)^T(y-x) < 0$$

since $\nabla_t z(t) \in \mathbb{R}^{n\times 1}$ and $\nabla_x f_0(x) \in \mathbb{R}^{n\times 1}$. Which means that $f_0(z(t)) < f_0(x)$ for small positive increase of t at $t = 0$. This implies that x is not the global minimum, and completes proof in forward direction.

- Reverse direction: recall the first order condition for convexity for function f

$$f(y) \geq f(x) + \nabla_x f(x)^T(y-x) \quad \forall y \in \mathcal{X}$$

If $\nabla_x f(x)^T(y-x) \geq 0$ for all $y \in \mathcal{X}$, then this implies that $f(y) \geq f(x)$ for all $y \in \mathcal{X}$, and shows that x is global minimum.

A geometric interpretation is illustrated in Figure 7.

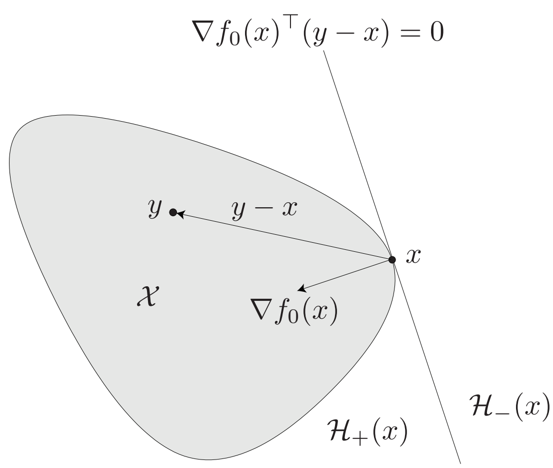

Figure 7: Geometric interpretation of an optimality condition. Here the shaded region represents the whole feasible set. Source: Boyd.

In other words, the optimal x^* locates the hyperplane formed by $\nabla f_0(x)$ such that all points $y \in \mathcal{X}$ lie on the side of the hyperplane such that the angle between $\nabla f_0(x)$ and $y - x$ is less than 90°.

This means that for all $y \in \mathcal{X} - \{x\}$, the objective increases and are thus not the optimal solution. In general, the direction of the gradient increases the objective and the opposite direction decreases the objective [7]. Thus, $x \in \mathcal{X}$ is as close to the global unconstrained minimum as it gets. In the case where the unconstrained global optimum is in the interior of $\mathcal{X}$, then $\nabla f_0(x) = 0$ at that point.

Example 2
Consider the convex optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_0(x) \\ \text{subject to:} \quad & x \succeq 0 \end{aligned}$$

where $x^* \neq 0$ is the global minimum. We can show that Theorem 4.1.2 follows from Theorem 4.1.3 in establishing global optimality of x^*.

If we pick

$$x^{(1)} = \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_i^* + 1 \\ \vdots \\ x_n^* \end{bmatrix}$$

Then $(x_i - x_i^*) \geq 0$ for all i and $\frac{\partial f_0(x^*)}{\partial x_i} \geq 0$ for all i in order to satisfy the global optimality condition in Theorem 4.1.3. If we pick

$$x^{(2)} = \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_i^*/2 \\ \vdots \\ x_n^* \end{bmatrix}$$

Then $(x_i - x_i^*) \leq 0$ for all i and $\frac{\partial f_0(x^*)}{\partial x_i} \leq 0$ for all i again in order to satisfy the global optimality condition in Theorem 4.1.3.

Since $x^{(1)}$ and $x^{(2)}$ are both in the feasible set $\mathcal{X}$, both conditions on $\frac{\partial f_0(x^*)}{\partial x_i}$ need to be satisfied, implying that $\nabla_x f_0(x) = 0$.

[7] As an illustration, if $f(x) = x^2$ is the objective, then $f'(x) = 2x$ is the gradient. When $x < 0$, $f'(x) < 0$, and decreasing x increases $f(x)$. However, changing x in the direction opposite of $f'(x)$ gives the direction that decreases the objective $f(x)$.

Theorem 4.1.4 The set of global minima points of an objective function $f_0 : \mathbb{R}^n \to \mathbb{R}$ includes the set

$$\{x^* \in \mathbb{R}^n \mid 0 \in \partial f_0(x^*)\}$$

Proof.

$$\begin{aligned} x^* \text{is global minimum point} &\Leftrightarrow f_0(x) \geq f_0(x^*) \quad \forall x \in \mathcal{X} \\ &\Leftrightarrow f_0(x) \geq f_0(x^*) + \mathbf{0}^T(x - x^*) \quad \forall x \in \mathcal{X} \\ &\Leftrightarrow 0 \in \partial f_0(x^*) \end{aligned}$$

To show why Theorem 4.1.3 is still necessary even with Theorem 4.1.2, consider the objective function $f_0(x) = |x|$ with dom $f_0 = \mathbb{R}$. In this case $\nabla_x f_0(x)$ is not defined at $x = 0$ even though $x = 0$ is a global minimum point.

4.2 Classes of Convex Optimization Problems

- **Linear Program (LP)**: minimization of a linear objective function subject to linear equality and inequality constraints. Conversion of a LP program of generic form to standard form is the operation

$$\begin{aligned} \min_x \quad & c^T x + d \\ \text{subject to:} \quad & Gx \leq h \\ & Ax = b \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \min_{\tilde{x}} \quad & \tilde{c}^T \tilde{x} \\ & \tilde{A}\tilde{x} = \tilde{b} \\ & \tilde{x} \geq 0 \end{aligned}$$

  Using the conversion tricks

  - $Gx \leq h \Leftrightarrow Gx + s = h$ for $s \geq 0$.
  - x can be rewritten as $x = x^+ - x^-$ with $x^+ \geq 0$ and $x^- \geq 0$.

  Example 1
  Minimization of the objective function $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$ can be written as a LP

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & a_i^T x + b_i \leq b \quad i = 1, \ldots, m \end{aligned}$$

  Example 2
  The optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & ||x||_1 \\ \text{s.t.} \quad & Gx \leq h \\ & Ax = b \end{aligned}$$

  can be converted into the LP

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n, t\in\mathbb{R}^n} \quad & t_1 + \cdots + t_n \\
\text{s.t.} \quad & Gx \le h \\
& Ax = b \\
& -t_i \le x_i \le t_i, \quad i = 1, \ldots, n
\end{aligned}
$$

- **Quadratic Program(QP)**: minimization of a quadratic function subject to linear equality and inequality constraints.

  A QP has the standard form.

$$
\begin{aligned}
\min_{x} \quad & \frac{1}{2} x^T P x + c^T x + d \\
\text{subject to} \quad & Gx \preceq h \\
& Ax = b
\end{aligned}
$$

  for $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, $G \in \mathbb{R}^{m\times n}$, $h \in \mathbb{R}^m$, $A \in \mathbb{R}^{p\times n}$, $b \in \mathbb{R}^p$, and $P \in \mathbb{S}^n_+$. Given a quadratic term $x^T Px$ in the quadratic objective, where P is not necessarily symmetric, one should first convert P to P' such that $P' = (P')^T$. Then the rest is determining whether P' is PSD. For example

$$
\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 5x_1x_2 + 4x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

  A QP is equivalent to LP when $P = 0$.

- **Quadratically Constrained Quadratic Program (QCQP)**: QP with quadratic inequality constraints. The standard form is

$$
\begin{aligned}
\min_{x} \quad & \frac{1}{2} x^T P_0 x + c_0^T x + d_0 \\
\text{subject to} \quad & \frac{1}{2} x^T P_i x + c_i^T x + d_i \le 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{aligned}
$$

  where in order for a QCQP to be convex, $P_0, P_1, \ldots, P_m$ need to be PSD. A QCQP becomes a QP if $P_i = 0$ for $i = 1, \ldots, m$.

  Example 3
  The constrained least squares problem

$$
\begin{aligned}
\min_{x} \quad & ||Ax - b||_2 \\
\text{s.t.} \quad & ||x||_2 \le 1
\end{aligned}
$$

  can be converted to the convex QCQP problem

$$
\begin{aligned}
\min_{x} \quad & ||Ax - b||_2^2 \\
\text{s.t.} \quad & ||x||_2^2 \le 1
\end{aligned}
$$

- **Second-Order Cone Program (SOCP)**: a convex optimization problem of the form

$$
\begin{aligned}
&\min_{x\in\mathbb{R}^n} c^T x \\
&||A_i x + b_i||_2 \le f_i^T x + d_i \quad i = 1,\dots,m \\
&Fx = g
\end{aligned}
$$

where the problem parameters are $c \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{m_i\times n}$, $b_i \in \mathbb{R}^{m_i}$, $f_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$, $F \in \mathbb{R}^{p\times n}$ and $g \in \mathbb{R}^p$. If $A_i = 0$, $b_i = 0$ for $i = 1,\dots,m$, then the problem becomes a linear programming problem. The existence of the term $||A_i x + b_i||_2$ covers many applications.

The origin of the name can be understood by considering the mathematical definition of a second-order cone

$$\mathcal{K}^n = \{(x,t) \in \mathbb{R}^n_+ \,:\, t \ge ||x||_2\}$$

which is graphically depicted as the cone in Figure 8.

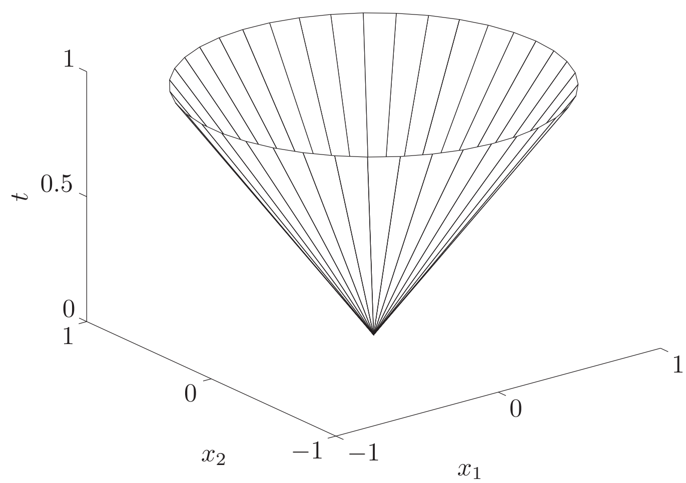

Figure 8: Second-order cone. Source: Boyd.

Example 4. Robust LP
A robust LP is a LP problem where the coefficients of the linear constraints are not known precisely. It takes on the form

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & c^T x \\
\text{s.t.} \quad & a_i^T x \le b_i, \quad i = 1,\dots,m \\
& a_i \in \xi_i = \{\bar{a}_i + P_i u \mid ||u||_2 \le 1\}
\end{aligned}
$$

where $P_1,\dots,P_m$ are invertible. In other words, the $a_1,\dots,a_m$ exist in an ellipsoid. The robust LP can be converted to a SOCP by adding the requirement that the constraints are satisfied regardless of choice of $a_i \in \xi_i$ for all i.

$$
\begin{aligned}
(\bar{a}_i + P_i u)^T x \le b_i \,&:\, ||u||_2 \le 1 \\
\Rightarrow (P_i u)^T x \le b_i - \bar{a}_i^T x \,&:\, ||u||_2 \le 1 \\
\max_{u:||u||_2\le 1} (P_i u)^T x \le b_i - \bar{a}_i^T x&
\end{aligned}
$$

The solution is $u^* = P_i^T x/||P_i^T x||_2$, which yields

$$||P_i^T x||_2 \le b_i - \bar{a}_i^T x$$

Thus, the SOCP form of robust LP that guarantees satisfied constraints are

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & ||P_i^T x||_2 \le b_i - \bar{a}_i^T x \quad i = 1, \dots, m \end{aligned}$$

- **Semidefinite Program (SDP)**: a SDP has the standard form

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0 \\ & Ax = b \end{aligned}$$

For symmetric matrices $F_0, F_1, \dots, F_n \in \mathbb{S}^k$. It is not important whether the constraint is $F_0 + x_1 F_1 + \cdots + x_n F_n \preceq 0$ or $F_0 + x_1 F_1 + \cdots + x_n F_n \succeq 0$, because setting $F_i^* = -F_i$ for all i maintains the symmetric requirement for F_i^*. This inequality is called a linear matrix inequality. The canonical form of SDP is

$$\begin{aligned} \min_{X\in\mathbb{R}^{n\times n}} \quad & \text{tr}(CX) \\ \text{s.t.} \quad & \text{tr}(A_i X) = b_i \quad i = 1, \dots, m \\ & X \succeq 0 \end{aligned}$$

where $X \in \mathbb{R}^{n\times n}$ and $C, A_1, \dots, A_m \in \mathbb{S}^n$.

5 Semidefinite Programs

Example problems such as

- Minimizing maximum eigenvalues of a matrix
- Minimizing sum of two largest eigenvalues
- Minimizing sum of all eigenvalues of a matrix
- Maximizing minimum eigenvalue of a matrix

can all be written as SDPs. Let $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \in \mathbb{S}^m$ for $x \in \mathbb{R}^n$ and symmetric matrices $A_0, \dots, A_n$. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$ be eigenvalues of $A(x)$.

- Minimizing maximum eigenvalue of $A(x)$ is the problem $\min_{x\in\mathbb{R}^n} \lambda_m(A(x))$, which can be expressed as a SDP using the epigraph trick

$$\begin{aligned}\min_{x\in\mathbb{R}^n, t\in\mathbb{R}} \quad & t\\ \text{s.t.} A(x) \preceq tI_m\end{aligned}$$

where the constraint ensures that the eigenvalues of $A(x)$ are bounded by t[8].

- Minimizing the sum of k greatest eigenvalues of a matrix is the problem $\min_{x\in\mathbb{R}^n}\sum_{i=k}^{m}\lambda_i(A(x))$, which can be converted to a SDP.

$$\begin{aligned}\min_{x\in\mathbb{R}^n, s\in\mathbb{R}, Z\in\mathbb{S}^m} \quad & \mathrm{tr}(Z)+s(m-k+1)\\ \text{s.t.} \quad & Z\succeq 0\\ & Z-A(x)+sI_m\succeq 0\end{aligned}$$

where

$$\begin{aligned}A(x) &= U\Lambda U^T\\ &= U\begin{bmatrix}\lambda_m-\lambda_k & & & 0\\ & \ddots & & \\ & & \lambda_k-\lambda_k & \\ 0 & & & 0\end{bmatrix}U^T + U\begin{bmatrix}0 & & & \\ & \lambda_{k-1}-\lambda_k & & \\ & & \ddots & \\ & & & \lambda_1-\lambda_k\end{bmatrix}U^T+\lambda_k I_m\\ &= UZU^T+UDU^T+\lambda_k I_m\end{aligned}$$

since D is negative semidefinite[9], then $A(x)-Z-\lambda_k I_m\preceq 0$.

$$\mathrm{tr}(Z)=\sum_{i=k}^{m}(\lambda_i-\lambda_k)=\left(\sum_{i=k}^{m}\lambda_i\right)-\lambda_k(m-k+1)$$

In the problem, let $s=\lambda_k$ since λ_k is unknown. Then the sum of k greatest eigenvalues is of the form $\mathrm{tr}(Z)+s(m-k+1)$, where s is to be solved subject to constraints.

- Maximizing the sum of k smallest eigenvalues of a matrix can be extended from the result above, since $\max_{x\in\mathbb{R}^n}\sum_{i=1}^{k}\lambda_i(A(x))$ can be rewritten as

$$\min_{x\in\mathbb{R}^n}\sum_{i=m-k+1}^{m}\lambda_i(B(x)),\quad B(x)=-A(x)$$

5.1 Max-cut

Max-cut is the problem of partitioning the vertices of a graph into two disjoint sets such that the number of edges between the two sets is maximum. The problem is hard because there are 2^{n-1} number of partitions with n nodes[10]. A more general problem occurs when each edge $e(i,j)$ has weight $w_{ij}\geq 0$, and the goal is to maximize $\sum w_{ij}$ for all $e(i,j)$ crossing the cut.

[8] To see this, let $A(X)=U\Lambda U^T$ by spectral theorem. Then $A(x)-tI_m=U(\Lambda-tI_m)U^T\preceq 0\Rightarrow\sum_{i=1}^{m}(\lambda_i-t)\tilde{u}_i^2$ for all $\tilde{u}=U^Tx\in\mathbb{R}^m$. This implies that $\lambda_i-t\leq 0$ for all i.

[9] Recall that $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_m$.

[10] To see why 2^{n-1}, imagine a binary decision tree for deciding whether to include node n_i in set A or B. Each leaf represents a final partition. A tree of height h has 2^h leaves, and since n decisions results in a tree of height $n-1$, the number of partitions is 2^{n-1}.

The optimization problem is formulated as follows. Let S_1 and S_2 be two partitions and let x_i be a variable associated with node i such that

$$x_i = \begin{cases} +1 & \text{if } x_i \in S_1 \\ -1 & \text{if } x_i \in S_2 \end{cases}$$

Let ξ be the edge set of the graph. Then the summation of weights belonging to edges crossing the cut is

$$\sum_{e(i,j)\in\xi} \frac{1}{2} w_{ij}(1 - x_i x_j) \tag{11}$$

This summation can be written as $x^T Lx$, where L is the Laplacian matrix for weighted graphs. The Laplacian matrix L is defined as

$$L_{ij} = \begin{cases} 0 & (i,j) \notin \xi \\ -w_{ij} & i \neq j, (i,j) \in \xi \\ \sum_{k:k\neq i} w_{ik} & i = j \end{cases}$$

Then

$$\begin{aligned} x^T Lx &= \sum_{i=1}^{n} y_i^2 \sum_{k\neq i} w_{ik} - 2 \sum_{e(i,j)\in\xi} x_i x_j w_{ij} \\ &= 2w(\xi) - 2 \sum_{e(i,j)\in\xi} x_i x_j w_{ij} \\ &= 4 \left(\sum_{e(i,j)\in\xi} w_{ij} \frac{1 - x_i x_j}{2} \right) \end{aligned} \tag{12}$$

where the coefficient 2 on the right hand side of the first equality comes from coefficients from $x^T Qx$ for $Q \in \mathbb{S}^n$. For the example where $n = 2$.

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & c \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2cx_1x_2 + dx_2^2$$

The term $2w(\xi)$ term on the right hand side of the second equality comes from the property graph property that the sum of degrees of every node in a graph is equal to $2|\xi|$ because each edge is counted twice. From the result of Equation 12 it can be concluded that $\frac{1}{4}x^T Lx$ is equivalent to Equation 11. Since the coefficient 1/4 will not affect the solution of max-cut, it can be dropped from the optimization problem formulation.

Thus the max-cut optimization problem is

$$
\begin{aligned}
\max_{x\in\mathbb{R}^n} \quad & x^T L x \\
\text{s.t.} \quad & x_i^2 - 1 = 0, \quad i = 1,\dots,n
\end{aligned} \tag{13}
$$

We can express $x^T Lx = \operatorname{tr}(x^T Lx) = \operatorname{tr}(Lxx^T) = \operatorname{tr}(LX)$, where the second equality comes from the trace property that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. The "almost" SDP formation of the optimization problem of Equation 13 is

$$
\begin{aligned}
\max_{X\in\mathbb{S}^n} \quad & \operatorname{tr}(QX) \\
\text{s.t.} \quad & X \succeq 0 \\
& X_{ii} = 1, \quad i = 1,\dots,n \\
& \operatorname{rank}(X) = 1
\end{aligned} \tag{14}
$$

However, the SDP formulation has no rank constraint, and removing the $\operatorname{rank}(X) = 1$ constraint in the optimization problem is called SDP relaxation.

Let x^* be the optimal solution of the original problem (13) and X^* the optimal solution of the relaxed problem (14) (without the rank 1 constraint). Since (14) is a relaxation, it is clear that:

$$\operatorname{tr}(QX^*) \geq x^{*\top} Q x^*$$

Now we seek to find an upper bound for the relaxed problem. That one is given by Goemans and Williamson in 1995[11].

Theorem 5.1.1 Let x^* be the optimal solution of the original problem (13) and X^* the optimal solution of the relaxed problem (14). Then:

$$\frac{1}{0.87} x^{*\top} Q x^* \geq \operatorname{tr}(QX^*) \geq x^{*\top} Q x^*$$

Proof: The right hand side inequality is trivial due to the relaxation of the SDP formulation. The proof for the left hand side can be found in Goemans and Williamson paper. The main idea is as follows. If X^* is not rank 1, then we know that $X^* \neq x^* x^{*\top}$. Define a probability distribution $Y \sim N(0, X^*)$, that is a multivariate gaussian with mean 0 and covariance X^*. Define a new probability distribution $\hat{x}$ as:

$$\hat{x}_i = \begin{cases} +1 & \text{if } y_i \geq 0, i = 1,\dots,k \\ -1 & \text{otherwise} \end{cases}$$

Then it is clear that:

$$\mathbb{E}(\hat{x}^\top Q \hat{x}) \leq x^{*\top} Q x^*$$

since:

$$\int \hat{x}^\top Q \hat{x} p_{\hat{x}}(\hat{x}) d\hat{x} \leq \int x^{*\top} Q x^* p_{\hat{x}}(\hat{x}) d\hat{x} = x^{*\top} Q x^*$$

and in addition, it can be showed that

$$\mathbb{E}(\hat{x}^\top Q \hat{x}) \geq 0.87 \cdot \operatorname{tr}(QX^*)$$

That implies that the Gaussian cut is 13% away from the optimal (in expectation).

[11] http://www-math.mit.edu/~goemans/PAPERS/maxcut-jacm.pdf

5.2 Polynomial Optimization

Let $p_0, p_1, \ldots, p_m$ be polynomials of any order. Polynomial optimization is a problem of the form

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & p_0(x) \\ \text{s.t.} \quad & p_i(x) \leq 0 \quad i = 1, \ldots, m \end{aligned}$$

Any polynomial optimization problem can be converted to a QCQP.

Example 1
The problem $\min_{x_1, x_2 \in \mathbb{R}} x_1^4 + x_1 x_2^2 + x_1^6$ can be converted to the following nonconvex QCQP.

$$\begin{aligned} \min_{x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}} \quad & x_3^2 + x_1 x_5 + x_3 x_4 \\ \text{s.t.} \quad & x_1^2 - x_3 = 0 \\ & x_3^2 - x_4 = 0 \\ & x_2^2 - x_5 = 0 \end{aligned}$$

The converted problem may not be convex because the quadratic constraints are not guaranteed to be convex.

6 Conic Optimization

A set $\mathcal{K} \subseteq \mathbb{R}^n$ is a cone if for all $x \in \mathcal{K}$ and all $\alpha > 0$, we have $\alpha x \in \mathcal{K}$. A proper cone $\mathcal{K}$ has the following properties.

- $\mathcal{K}$ is convex: for all $x, y \in \mathcal{K}$, $\alpha_1 x + \alpha_2 y \in \mathcal{K}$ for all $\alpha_1, \alpha_2 > 0$.
- $\mathcal{K}$ is closed.
- $\mathcal{K}$ is solid: $\mathcal{K}$ has a nonempty interior. For example, a line in $\mathbb{R}^2$ is not a proper cone (since it does not have interior).
- $\mathcal{K}$ is pointed: if $x \in \mathcal{K}$ and $-x \in \mathcal{K}$, then $x = 0$.

For example the set $\mathbb{R}^2_+$ is a proper cone. The half place $\{x \in \mathbb{R}^2 : x_2 \geq 0\}$ is not a proper cone because it is not pointed. For a proper cone $\mathcal{K}$.

- $x \succeq_{\mathcal{K}} y \Leftrightarrow x - y \in \mathcal{K}$
- $x \succ_{\mathcal{K}} y \Leftrightarrow x - y \in \text{int}\mathcal{K}$

A conic linear optimization problem has the following standard form

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & c^T x \\ \text{s.t.} \quad & A_i x - b_i \preceq_{\mathcal{K}_i} 0 \quad i = 1, \ldots, m \\ & Fx = g \end{aligned}$$

If $\mathcal{K} = \mathbb{R}^n_+$, then the conic optimization problem becomes a LP. Let X be defined as

$$X = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \end{bmatrix}$$

where $x^{(i)} \in \mathbb{R}^n$. Then $X \succeq_{\mathcal{K}} 0$ for $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots$ is equivalent to[12]

$$\begin{aligned} x^{(1)} &\succeq_{\mathcal{K}_1} 0 \\ x^{(2)} &\succeq_{\mathcal{K}_2} 0 \\ &\vdots \end{aligned}$$

7 Separating and Supporting Hyperplanes

7.1 Separating Hyperplane Theorem

Suppose C and D are two convex sets that do not intersect, i.e. $C \cap D = \emptyset$. Then $\exists a \neq 0$ and b such that:

$$\begin{aligned} a^\top x \leq b, \ \forall x \in C \\ a^\top x \geq b, \ \forall x \in D \end{aligned}$$

[12]Let C_1 and C_2 be two sets, then $C_1 \times C_2$ is the set of ordered pairs $\{(c_1, c_2) | c_1 \in C_1, c_2 \in C_2\}$.

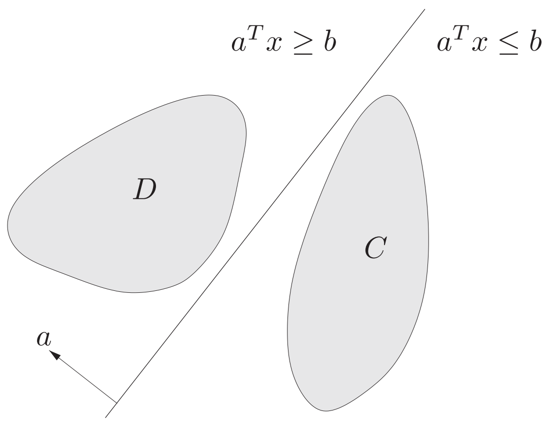

Figure 9: Separating hyperplane for C and D. Source: Boyd.

Note that the inequalities are not always strict. As an example consider the sets in $\mathbb{R}^2$ defined by $C = \{(x, y) \ : \ y \leq 0\}$ and $D = \{(x, y) \ : \ y \geq e^x\}$. The only possible separating hyperplane is given by $y = 0$, which is in the set C.

Sketch of the proof:

1. Find (c, d) as the points that minimize the distance between the sets
$$\text{dist}(C, D) = \inf\{||u - v||_2 \ : \ u \in C, v \in D\}$$
2. Find the midpoint of the segment
3. Draw an orthogonal hyperplane going through the midpoint (that is orthogonal to the segment that connects c and d). For that, define:
$$a = d - c, \qquad b = \frac{||d||_2 - ||c||_2}{2}$$
then $a^\top x = b$ is a separating hyperplane.

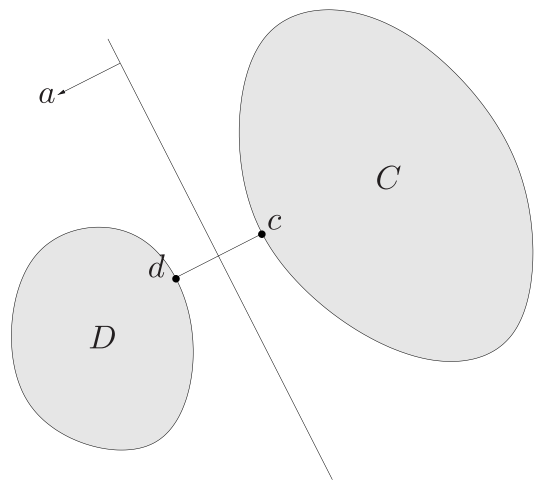

Figure 10: Construction of the separating hyperplane. Source: Boyd.

> **Theorem 6.1** Let $C \cap D = \emptyset$. If C is a single point and D is a closed set, then there is strict separation.

Figure 11: On the left, D is closed and so we can ensure strict separation. On the right, since D is open, there may be cases on which we cannot ensure strict separation.

7.2 Converse Separating Hyperplane Theorem

Any convex set C and D, with at least one being open, are disjoint if and only if there is a separating hyperplane.

Note that the existence of the hyperplane does not imply that the sets are disjoint, as demonstrated in Figure 12.

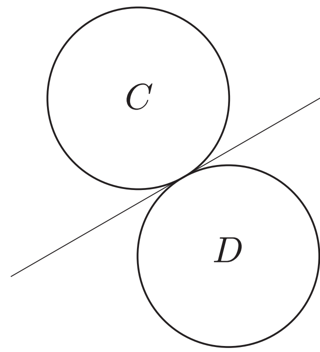

Figure 12: A sufficient condition is that one set should be open. In this case, since both sets are closed, we can find a separating hyperplane, but they are not disjoint.

7.3 Theorem of Alternatives for Strict Inequalities

We want to find necessary and sufficient conditions on $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ such that the inequality $Ax < b$ has no solutions.

Define $C = \{b - Ax \ : \ x \in \mathbb{R}^n\}$ and $D = \mathbb{R}^n_{++}$. Observe that the original problem $Ax < b$ is infeasible if and only if $C \cap D = \emptyset$. Also, note that C and D are convex and D is open. So, by the converse separating hyperplane theorem:

$$\begin{aligned} C \cap D = \emptyset \ \Leftrightarrow \ & \exists \lambda \in \mathbb{R}^m, \mu \in \mathbb{R} \\ & \text{s.t. } \begin{cases} \lambda^\top y \le \mu & \forall y \in C \\ \lambda^\top y \ge \mu & \forall y \in D \end{cases} \end{aligned}$$

The previous conditions can be simplified. The first one implies that $\lambda^\top (b - Ax) \le \mu$ for all x. This can be written as $\lambda^\top b - \lambda^\top Ax \le \mu$ for all x. Since $\lambda^\top Ax$ could reach ∞ if x is unrestricted, the only condition that guarantees the inequality for all x is when $\lambda^\top A = 0$. This implies that $\lambda^\top b \le \mu$

From the second inequality we got that $\lambda^\top y \ge \mu$ for all $y > 0$. For $y \to 0$ we obtain that $\lambda^\top y \approx 0 \ge \mu$ and so $\mu \le 0$. Since μ is nonpositive and $y > 0$ we require that $\lambda \ge 0$ and $\lambda \ne 0$. This implies that $\lambda^\top b \le \mu \le 0$. Putting all the conditions together we obtain:

$$\exists \lambda \in \mathbb{R}^m \text{ s.t. } \lambda \neq 0, \ \lambda \ge 0, \ A^\top \lambda = 0, \ \lambda^\top b \le 0 \tag{15}$$

In summary, the set of inequalities $Ax < b$ is infeasible if the set of inequalities: $\lambda \ge 0, A^\top \lambda = 0, \lambda^\top b \le 0$ has a nonzero solution. This means that $Ax < b$ and (15) form a pair of alternatives: for any data A and b, exactly one of them is solvable.

7.4 Supporting hyperplane

If C is a closed convex set, and D is a single point x_0 on the boundary of C, then there is a separating hyperplane which is called supporting hyperplane:

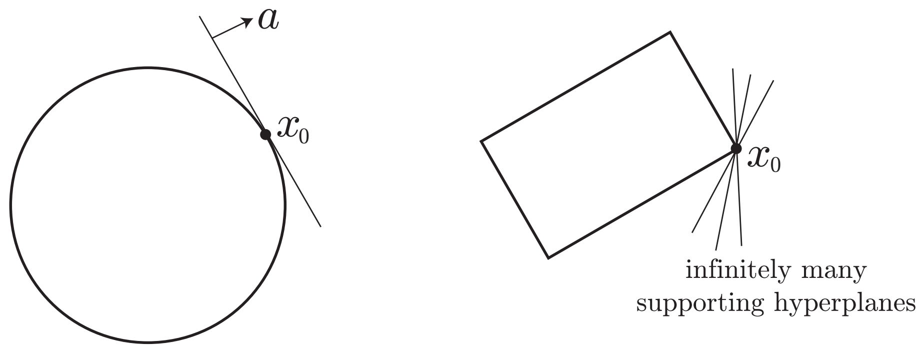

Figure 13: On the left, a smooth point has a unique and tangential supporting hyperplane. On the right, a sharp point admits infinitely many supporting hyperplanes.

A point may have infinitely many supporting hyperplanes, but at a smooth point x_0 (in simple terms, that is not a corner of two linear constraints) separating hyperplane is unique and tangential to the set.

7.5 Dual Cones

Given a cone K, the dual cone K^* is defined as

$$K^* = \{y \mid \langle x, y \rangle \geq 0, \ \forall x \in K\} \tag{16}$$

K^* is always a cone, even if K is not.

Geometrically, $y \in K^*$ if and only if $-y$ is the normal of a hyperplane that supports K at the origin as illustrated in the following figure

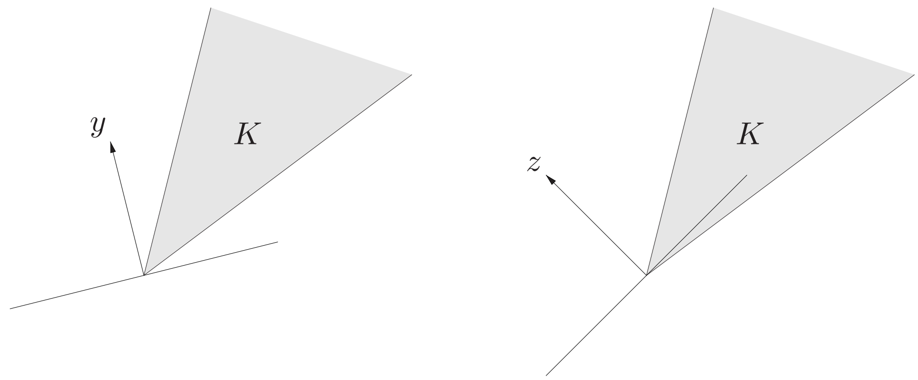

Figure 14: On the left, the halfspace with inward normal y contains the cone K and so $y \in K^*$. While on the right, the halfspace with normal z does not contain K and so $z \notin K^*$. Source: Boyd.

From the definition of a dual cone it is clear that if $y \in K^*$ and $x \in K$ the angle between x and y is less than 90 degrees.

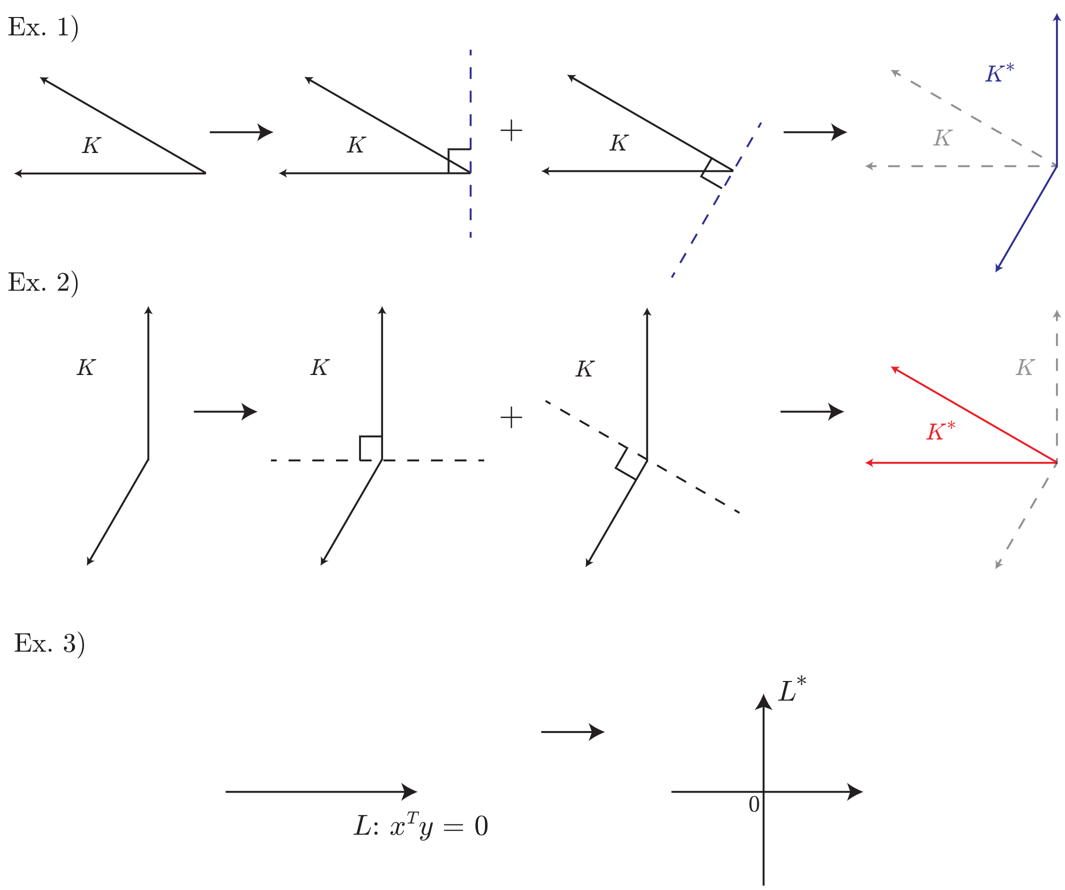

Figure 15: Examples of dual cones construction using the orthogonal vectors of the original cone.

A few examples of dual cones are

- If $K = \mathbb{R}^n_+$ then $K^* = K$.
- If K is a second order cone, then $K^* = K$
- If K is a PSD cone, then $K^* = K$.
- If K is a proper cone, then $K^{**} = K$.

For a PSD cone $\mathcal{K}$, then its dual cone is defined as the set

$$\mathcal{K}^* = \{Y \mid \text{Tr}(XY) \geq 0, \forall X \in \mathcal{K}\}$$

Example 1
We will prove the second point that if K is a second order cone, then $K^* = K$. Let $\mathcal{K}_{soc} \in \mathbb{R}^n$ denote a second-order cone. In other words

$$\mathcal{K}^n = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}_+ \,:\, t \geq ||x||_2\}$$

Another way to denote a second-order cone is with a vector $x \in \mathbb{R}^n$ such that $x_n \geq ||x_1^{n-1}||_2$. Here we let $x_1^k \in \mathbb{R}^k$ denote the vector with entries $x_1, \dots, x_k$ from $x \in \mathbb{R}^n$.

Showing that $\mathcal{K}_{soc} \subseteq \mathcal{K}_{soc}^*$ and $\mathcal{K}_{soc}^* \subseteq \mathcal{K}_{soc}$, shows that $\mathcal{K}_{soc}^* = \mathcal{K}_{soc}$ exactly.

Show $\mathcal{K}_{soc} \subseteq \mathcal{K}_{soc}^*$. Let $s, x \in \mathcal{K}_{soc}$. Then

$$s^\top x = s_n x_n - \sum_{i=1}^{n-1} s_i x_i \geq s_n x_n - ||s_1^{n-1}||_2 ||x_1^{n-1}||_2 \geq 0$$

where the first inequality comes from Cauchy-Schwartz inequality and the second inequality follows from assumption that $s, x \in \mathcal{K}_{soc}$. Since this is true for any $x \in \mathcal{K}_{soc}$, this shows that $s \in \mathcal{K}_{soc}^*$.

Show $\mathcal{K}_{soc}^* \subseteq \mathcal{K}_{soc}$. Assume $s \in \mathcal{K}_{soc}^*$ and $x \in \mathcal{K}_{soc}$. There are two cases to consider - (1) when $x_1, \dots, x_{n-1} = 0$ and $x_n = a > 0$ and (2) when $x_1, \dots, x_{n-1} \neq 0$ with $x_n = t \geq ||x_1^{n-1}||_2$.

- Case 1: construct s such that $s_1, \dots, s_{n-1} = 0$. Then
$$s^\top x \geq 0 \Leftrightarrow s_n a \geq 0 \Leftrightarrow s_n \geq ||s_1^{n-1}||_2 \Leftrightarrow s \in \mathcal{K}_{soc}$$
- Case 2: construct s such that $s_i = -x_i$ for $i = 1, \dots, n-1$. Then
$$s^\top x = s_n t - \sum_{i=1}^{n-1} s_i^2 = s_n t - ||s_1^{n-1}||_2^2 \geq 0 \Leftrightarrow s_n \geq \frac{||s_1^{n-1}||_2}{t}$$
Then set $s_n \geq \frac{||s_1^{n-1}||_2}{t}$ to finally create s such that $s^T x \geq 0$ for all x satisfying case 2 conditions. Since $s_n \geq ||s_1^{n-1}||_2$ by construction, $s \in \mathcal{K}_{soc}$.

7.6 Theorem of Alternatives for Linear Strict Generalized Inequalities

Assume K is a proper cone, then $\nexists\, x \in \mathbb{R}^n$ such that $Ax \preceq_K b$ if and only if:

$$\exists \lambda \neq 0,\ \lambda \succeq_{K^*} 0,\ A^\top \lambda = 0,\ \lambda^\top \leq 0$$

This is a general version of the theorem of alternatives for strict inequalities. The proof is similar, requiring the affine set $C = \{b - Ax \ :\ x \in \mathbb{R}^n\}$ and $D = \mathrm{int}(K)$. The condition of infeasibility implies that those sets do not intersect and so a separating hyperplane can be found.

8 Duality

Consider the following generic, possibly non-convex, optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p \end{aligned} \tag{17}$$

with optimal objective value p^*. We associate a scalar variable to each constraint:

$$\begin{aligned} f_i(x) \leq 0 \ \leftarrow \ \lambda_i \\ h_j(x) = 0 \ \leftarrow \ \nu_j \end{aligned}$$

We define the **Lagrangian** as $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ as

$$\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)$$

with $\mathrm{dom}(\mathcal{L}) = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p$, where $\mathcal{D}$ is the original domain of the optimization problem and λ_i's and ν_j's are called the **Lagrange multipliers** or **dual parameters**.

Theorem 7.1 For an optimization problem in the standard form in Equation 17, as long as $\lambda_i \geq 0$ for $i = 1, \ldots, m$, then

$$g(\lambda, \nu) \leq p^*$$

where $g(\lambda, \nu) = \inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda, \nu)$.

A direct consequence of this is that $d^* = \max_{\lambda \geq 0, \nu} g(\lambda, \nu) \leq p^*$.

Proof.
Let x^* denote the optimal solution to the original problem in Equation 17.

$$\begin{aligned} f_0(x^*) &\geq f_0(x^*) + \sum_{i=1}^{m} \underbrace{\lambda_i}_{\geq 0} \underbrace{f_i(x^*)}_{\leq 0} + \sum_{j=1}^{p} \underbrace{\nu_j h_j(x^*)}_{=0} \\ &\geq \min_x \mathcal{L}(x, \lambda, \nu) = g(\lambda, \nu) \end{aligned}$$

where the second inequality is due to the fact that x^* does not necessarily equal $\arg\max_x \mathcal{L}(x, \lambda, \nu)$.

Theorem 7.2 $-g(\lambda, \nu)$ is always convex for arbitrary functions f_i's and h_j's.

Proof.
$g(\lambda, \nu) = \min_x \mathcal{L}(x, \lambda, \nu)$ is the pointwise minimum of affine functions in λ and ν. Since an affine function is both convex and concave, and the pointwise minimum of concave functions is concave. A function f is concave if and only if the set $\{(x, y) : y \leq f(x)\}$ is convex. The pointwise minimum of concave functions $f(x) = \min_i f_i(x)$ results in the set $\{(x, y) : y \leq f(x)\}$ that is the intersection of the sets $\{(x, y) : y \leq f_i(x)\}$ for all i. Since the intersection of convex sets is convex, $\{(x, y) : y \leq f(x)\}$ is convex, implying that $f(x)$ is concave. The negative of a concave function is convex.

The dual solution is found as $d^* = \max\limits_{\lambda \geq 0, \nu} g(\lambda, \nu)$.

Theorem 7.3 If weak duality holds

$$p^* \geq d^*$$

Theorem 7.4 The dual problem is always convex even if its associated primal is not convex.

Proof.
From Theorem 7.2, $-g(\lambda,\nu)$ is always convex. The dual problem is

$$\begin{aligned} \max_{\lambda} \quad & g(\lambda,\nu) \\ & \lambda \geq 0 \end{aligned}$$

which can be written as a convex optimization problem in standard form

$$\begin{aligned} \min_{\lambda} \quad & -g(\lambda,\nu) \\ & \lambda \geq 0 \end{aligned}$$

Example 1
Find the dual problem to a LP. The standard LP problem is stated as

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & c^\top x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

The Lagrangian is

$$\mathcal{L}(x,\lambda,\nu) = c^\top x - \sum_{i=1}^{m} \lambda_i x_i + \nu^\top (Ax - b) = -b^\top \nu + (c + A^\top \nu - \lambda)^\top x$$

The dual problem is then found as

$$g(\lambda,\nu) = \inf_x \mathcal{L}(x,\lambda,\nu) = -b^\top \nu + \inf_x (c + A^\top \nu - \lambda)^\top x$$

which leads to the dual problem

$$g(\lambda,\nu) = \begin{cases} -b^\top \nu & \text{if } c + A^\top \nu - \lambda = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Maximizing the dual problem is another LP

$$\begin{aligned} \min_{\nu,\lambda} \quad & -b^\top \nu \\ & c + A^\top \nu - \lambda = 0 \\ & \lambda \geq 0 \end{aligned}$$

Example 2
We start by reformulating Equation 13 in the max-cut problem into standard form

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & x^T Q x \\
\text{s.t.} \quad & x_i^2 - 1 = 0, \quad i = 1,\dots,n
\end{aligned} \tag{18}
$$

where $Q = -L$. The Lagrangian is

$$
\mathcal{L}(x,\nu) = x^\top Q x + \sum_{i=1}^n \nu_i(x_i^2 - 1) = x^\top \left(Q + \begin{bmatrix} \nu_1 & & & 0 \\ & \nu_2 & & \\ & & \ddots & \\ 0 & & & \nu_n \end{bmatrix} \right) x - \sum_{i=1}^n \nu_i
$$

which has the dual objective

$$
g(\nu) = \begin{cases} -\sum_{i=1}^n \nu_i & \text{if } Q + \begin{bmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_n \end{bmatrix} \succeq 0 \\ -\infty & \text{otherwise} \end{cases}
$$

Example 3
Find the dual problem of a QP. One expression of a QP is

$$
\begin{aligned}
\min_x \quad & \frac{1}{2} x^\top P x + q^\top x + d \\
\text{s.t.} \quad & Ax \preceq b \\
& Cx = h
\end{aligned}
$$

Assume without loss of generality that $C = 0$, $q = 0$, and $d = 0$. The Lagrangian is

$$
\mathcal{L}(x,\lambda) = \frac{1}{2} x^\top Q x + \lambda^\top (Ax - b)
$$

Solving for x^*

$$
\begin{aligned}
\nabla_x \mathcal{L}(x,\lambda) &= Px + A^\top \lambda = 0 \\
&\Rightarrow x^* = -P^{-1} A^\top \lambda
\end{aligned}
$$

The dual objective then becomes

$$
g(\lambda) = -\frac{1}{2} \lambda^\top A P^{-1} A^\top \lambda - \lambda^\top b
$$

which is a quadratic objective. Thus the dual problem of a QP is another QP.

Exercise 1
Show that the dual of a QCQP is a SOCP.

Exercise 2

Show that the dual of a SOCP is another SOCP.

Solution:
Consider a SOCP in standard form:

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & ||A_i x + b_i||_2 \leq c_i^\top + d_i, \ i = 1, \ldots, m \end{aligned}$$

We have:

$$\begin{aligned} p^* &= \min_x \max_{\lambda \geq 0} c^\top x + \sum_{i=1}^m \lambda_i (||A_i x + b_i||_2 - c_i^\top - d_i) \\ &= \min_x \max_{\lambda \geq 0} \max_{||u_i||_2 \leq 1, i=1,\ldots,m} \quad c^\top x + \sum_{i=1}^m \lambda_i [u_i^\top (A_i x + b_i) - c_i^\top - d_i] \\ &= \min_x \max_{||u_i||_2 \leq \lambda_i, i=1,\ldots,m} \quad c^\top x + \sum_{i=1}^m u_i^\top (A_i x + b_i) - \lambda_i (c_i^\top x + d_i) \end{aligned}$$

on which, in the second line, we use the dual norm representation of the 2-norm. We then compute the dual problem as:

$$\begin{aligned} d^* = & \max_{||u_i||_2 \leq \lambda_i, i=1,\ldots,m} \quad \min_x c^\top x + \sum_{i=1}^m u_i^\top (A_i x + b_i) - \lambda_i (c_i^\top x + d_i) \\ = & \max_{||u_i||_2 \leq \lambda_i, i=1,\ldots,m} \sum_{i=1}^m \left(u_i^\top b_i - \lambda_i d_i \right) + \min_x x^\top \left(c + \sum_{i=1}^m A_i^\top u_i - \lambda_i c_i \right) \\ = & \max_{u,\lambda} \quad \sum_{i=1}^m \left(u_i^\top b_i - \lambda_i d_i \right) \\ & \quad \text{s.t.} \ ||u_i||_2 \leq \lambda_i, \quad i = 1, \ldots, m \\ & \quad \quad c + \sum_{i=1}^m \left(A_i^\top u_i - \lambda_i c_i \right) = 0 \end{aligned}$$

that is an SOCP.

Recall the conjugate function defined in Equation 4. The conjugate function and dual objective are closely related. Consider the generic optimization problem with linear inequality and equality constraints.

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & Ax \preceq b \\ & Cx = d \end{aligned}$$

The dual objective is

$$
\begin{aligned}
g(\lambda, \nu) &= \min_x f_0(x) + (A^\top \lambda + C^\top \nu)^\top x + (-\lambda^\top b - \nu^\top d) \\
&= -\max_x \Big((-A^\top \lambda - C^\top \nu)^\top x - f_0(x) \Big) + (-\lambda^\top b - \nu^\top d) \\
&= -f_0^*(-A^\top \lambda - C^\top \nu) + (-\lambda^\top b - \nu^\top d)
\end{aligned}
$$

Thus, the dual optimization problem can be written in terms of the conjugate function when the constraints are linear.

$$
\begin{aligned}
\max_{\lambda, \nu} \quad & -f_0^*(-A^\top \lambda - C^\top \nu) + (-\lambda^\top b - \nu^\top d) \\
& \lambda \geq 0
\end{aligned}
\tag{19}
$$

The domain of g follows from the domain of f_0^*

$$
\text{dom } g = \{(\lambda, \nu) \mid -A^\top \lambda - C^\top \nu \in \text{dom } f_0^*\}
$$

Example 4. Recall from Example 2 of section 3.1 that the conjugate of $f(x) = ||x||$ is

$$
f^*(y) = \begin{cases} 0 & ||y||_* \leq 1 \\ \infty & \text{otherwise} \end{cases}
$$

The dual problem of the primal minimization problem

$$
\begin{aligned}
\min_x \quad & ||x|| \\
& Ax = b
\end{aligned}
$$

can be rewritten in terms of the conjugate of $||x||$ by Equation 19 as

$$
\begin{aligned}
\max_\nu \quad & -f_0^*(-A^\top \nu) - b^\top \nu \\
& ||-A^\top \nu||_* \leq 1
\end{aligned}
$$

Duality gap. Let p^* be the optimal value of the primal problem and d^* the optimal value of its dual. We define the duality gap as

$$
\text{duality gap} = p^* - d^*
$$

When duality gap is equal to zero, we say that strong duality holds.

Definition. Consider an optimization problem in standard form as

$$
\begin{aligned}
\min_x \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m
\end{aligned}
$$

with no convexity assumption and associated dual multipliers λ_i. We say $\lambda^* \geq 0$ is called a **geometric multiplier** if

$$p^* = \min_x \mathcal{L}(x, \lambda^*)$$

8.1 Geometric Interpretation of Duality

For simplicity we consider an optimization problem with one constraint

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_1(x) \leq 0, \end{aligned}$$

Consider the region in the (u, t) plane defined by:

$$\mathcal{G} = \{f_1(x), f_0(x)\}$$

evaluating over x, $\mathcal{G}$ defines a region in the (u, t) space. The optimal value $p^\star$ can be found by finding the minimum point projected over the t axis, that satisfies $u \leq 0$ (that is equivalent as $f_1(x) \leq 0$). This is depicted in the following figure

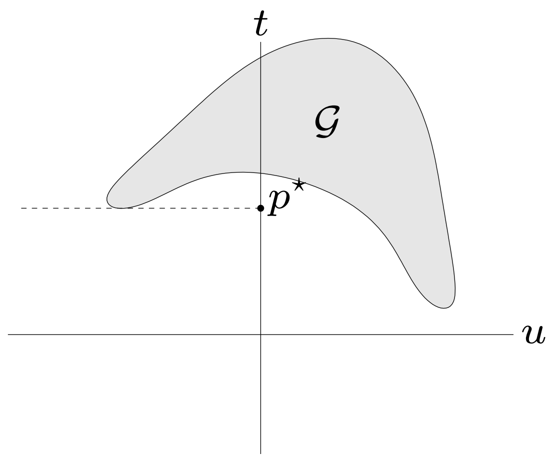

Figure 16: Optimal value $p^\star$ ($t = f_0(x^\star)$) in the (u, t) space.

Now, recall that the dual function is defined as

$$g(\lambda) = \min_x f_0(x) + \lambda f_1(x) = \min_{(u,t) \in \mathcal{G}} \ t + \lambda u$$

for a given $\lambda \geq 0$. For an α not necessary equal to $g(\lambda)$ and a given $\lambda \geq 0$, $\alpha = t + \lambda u \Rightarrow t = \alpha - \lambda u$ specifies a line in (u, t) space. Geometrically, the search for $g(\lambda)$ can be thought of as searching for the minimum α such that the line $t = \alpha - \lambda u$ still touches $\mathcal{G}$, since there must be a x that such (u, t) is possible. This concept is illustrated in Figure 17.

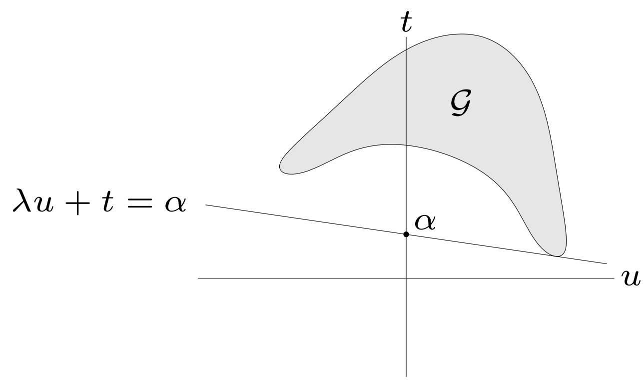

Figure 17: Interpretation of line $t + \lambda u = \alpha$.

By definition, $d^* = \max_{\lambda \geq 0} \min_{(u,t) \in \mathcal{G}} t + \lambda u$. Once $\alpha_1, \ldots, \alpha_n, \ldots$ and their corresponding lines are determined, d^* is found as $\max_i \alpha_i$ and the solution is the corresponding $\lambda \geq 0$.

This concept is depicted in Figure 18, where the blue line shows the best supporting hyperplane that achieves $d^\star$. Even though the red line achieves a better intersection on the t-axis, it will not be among the lines considered for d^* because the line can achieve a lower intercept by translating further in the southwest direction. Black lines show other valid supporting hyperplanes, but that do not achieve $d^\star$.

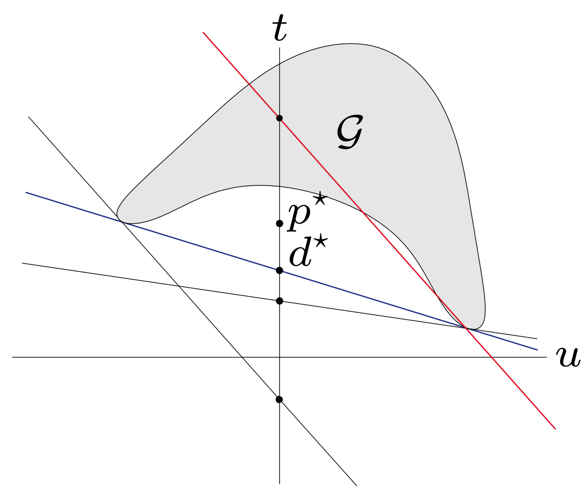

Figure 18: Finding the optimal value for the dual problem. The blue line achieves $d^\star$, while the red line is not a valid supporting hyperplane. As can be observed, there exists a duality gap $p^\star - d^\star$ in this problem.

Theorem 7.1.1 If there is no duality gap, the set of geometric multipliers is equal to the set of optimal dual solutions:

$$\lambda^* = \arg\max_{\lambda \geq 0} \; g(\lambda)$$

Theorem 7.1.2 If there is a duality gap, even though the set of dual multipliers may not be empty, the set of geometric multipliers is empty.

8.2 Strong Duality

Consider a generic optimization problem

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \end{aligned}$$

Assume $\lambda^* \geq 0$ is a geometric multiplier, then

$$\begin{aligned} p^* &= f_0(x^*) \\ &= \min_x \mathcal{L}(x, \lambda^*) \\ &= \min_x f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) \\ &\leq f_0(x^*) + \sum_{i=1}^m \underbrace{\lambda_i^*}_{\geq 0} \underbrace{f_i(x^*)}_{\leq 0} \\ &\leq f_0(x^*) \end{aligned}$$

where the first equality follows from definition of primal optimal value, the second and third equalities follow from the definition of geometric multiplier and Lagrangian respectively, the inequality on the fourth line follows from the fact that possibly $x^* \neq \arg\min_x \mathcal{L}(x, \lambda^*)$, and the inequality of the fifth line follows from the fact $\lambda_i^* \geq 0$ and $f_i(x^*) \leq 0$ for $i = 1, \ldots, m$. However, since $p^* = f_0(x^*)$, this implies that

$$\lambda_i^* f_i(x^*) = 0, \quad i = 1, \ldots, m$$

and this is a condition called **Complementary Slackness**.

Theorem 7.2.1 The existence of a geometric multiplier implies no duality gap.

Proof

1. Weak duality implies $p^* \geq d^*$.
2. $p^* = \min_x \mathcal{L}(x, \lambda^*) \leq \max_{\lambda \geq 0} \min_x \mathcal{L}(x, \lambda) = d^*$
   To see why this is true, notice that $\max_{\lambda \geq 0} \min_x \mathcal{L}(x, \lambda) < \min_x \mathcal{L}(x, \lambda^*)$ cannot be true. Assume the strict inequality is true and let $x^\circ = \arg\min_x \mathcal{L}(x, \lambda^*)$. Building off of the geometric interpretation of duality, the solution of $\max_{\lambda \geq 0} \min_x \mathcal{L}(x, \lambda)$ can be interpreted as $\arg\max_i \min_x \mathcal{L}(x, \lambda_i)$. Thus, at least for $\lambda_i = \lambda^*$, the inner minimization would pick $x = x^\circ$, which contradicts the assumption.

Thus, $p^* = d^*$.

8.3 Necessary and Sufficient Conditions for Zero Duality Gap

Consider the point (x^*, λ^*) that exists and satisfies

(1) Primal feasibility: $f_i(x^*) \leq 0,\ i = 1, \ldots, m$

(2) Dual feasibility: $\lambda_i^* \geq 0,\ i = 1, \ldots, m$.

(3) Complementary slackness: $\lambda_i^* f_i(x^*) = 0,\ i = 1, \ldots, m$

(4) Lagrangian minimization: $\mathcal{L}(x^*, \lambda^*) = f_0(x^*) = \min_x \mathcal{L}(x, \lambda^*)$.

Then, there is no duality gap. If equality constraints exist then $h_j(x)$ for $j = 1, \ldots, p$ are included accordingly.

Recall that condition 4 implies condition 3, but the converse is not true. The converse is not true because x^* satisfying complementary slackness alone does not guarantee that $f(x^*) = p^*$. Observe that checking condition (4) is as hard as solving the primal problem itself. A necessary condition for (4) is given by:

(4′) Lagrangian Stationarity condition:

$$\nabla \mathcal{L}(x, \lambda^*)\Big|_{x=x^*} = 0$$

$$\Rightarrow \nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) = 0$$

Conditions (1), (2), (3) and (4′) are also called **Karush-Kuhn-Tucker (KKT) conditions**. For any optimization problem where strong duality holds, the KKT conditions are necessary for optimal primal-dual pair (x^*, λ^*). If the problem is convex under constraint qualifications, then the KKT conditions are sufficient to establish optimality of primal-dual pair (x^*, λ^*).

8.4 Local Behaviour

Consider an optimization problem with equality constraints

$$\begin{aligned} \min_{x} \quad & f_0(x) \\ \text{s.t.} \quad & h_i(x) = 0, \qquad i = 1, \ldots, p \end{aligned}$$

We want to find a local solution, and for that purpose we will use local analysis. Consider the point x^*, we are interested in studying the conditions under which x^* is a local minimum, by analyzing the feasible set around x^* given by its tangent plane. Some examples of tangent planes are depicted on the following figures.

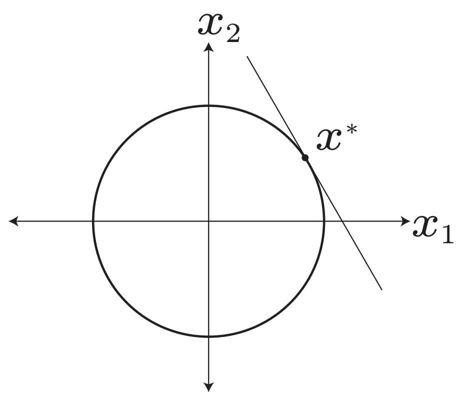

Figure 19: Tangent plane for $h(x) = x_1^2 + x_2^2 - 1$.

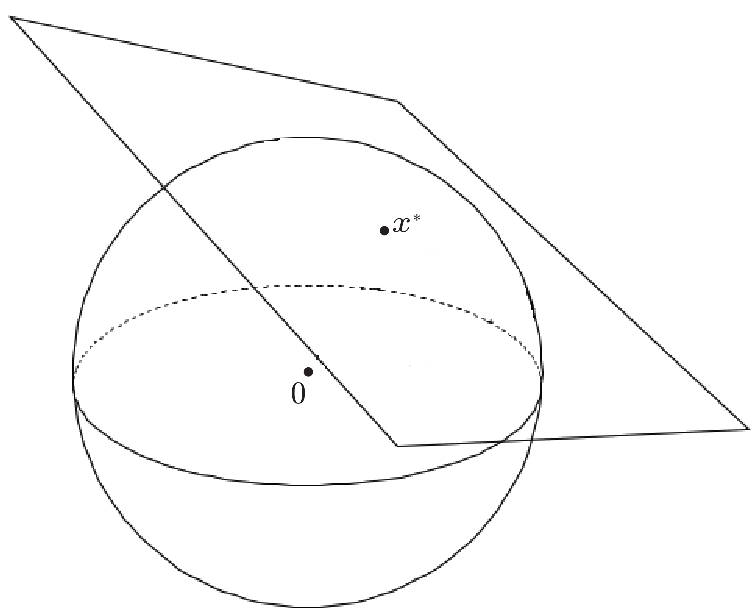

Figure 20: Tangent plane for $h(x) = x_1^2 + x_2^2 + x_3^2 - 1$.

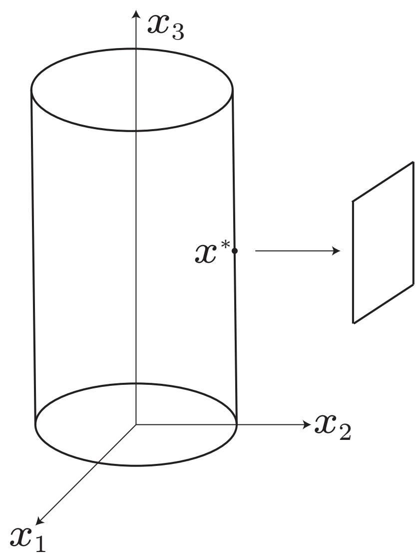

Figure 21: Tangent plane for $h(x) = x_1^2 + x_2^2 - 1, x \in \mathbb{R}^3$.

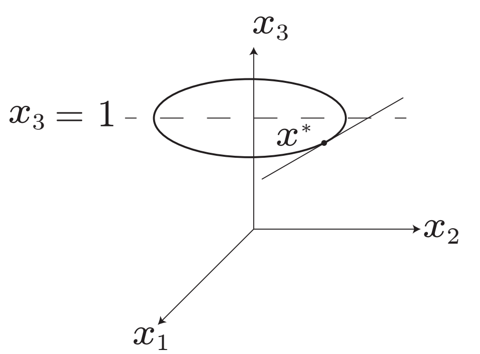

Figure 22: Tangent plane for $h_1(x) = x_1^2 + x_2^2 - 1$ and $h_2(x) = x_3 - 1$.

How to find a tangent plane?

Definition. A point x^* is called **regular** if vectors $\nabla h_1(x^*), \nabla h_2(x^*), \ldots, \nabla h_p(x^*)$ are linearly independent at this single point. Briefly, vectors $v_1, \ldots v_n$ are linearly dependent if there exists a set of coefficients $a_1, \ldots, a_n$ where not all of them is zero, such that $a_1 v_1 + \cdots + a_n v_n = 0$ is possible.

Definition. If x^* is regular, then the **tangent plane** of feasible set at x^* is given by

$$\left\{\Delta x \in \mathbb{R}^n \;:\; \nabla h_i(x^*)^\top \Delta x = 0,\; i = 1, \ldots, p\right\}$$

Where Δx can be viewed as a small perturbation. This comes from the Taylor approximation near x^*

$$h_i(x^* + \Delta x) = h_i(x^*) + \nabla h_i(x^*)^\top \Delta x + \text{higher order terms}$$

since $h_i(x^*) = 0$, and we want $h_i(x^* + \Delta x) \to 0$ near x^*, then we require $\nabla h_i(x^*)^\top \Delta x = 0$.

Example 1. The tangent plane for $h(x) = x_1^2 + x_2^2 + x_3^2 - 1$ is given by $\nabla h(x)^\top = (2x_1, 2x_2, 2x_3)$.

Following definition of linear independence mentioned above, the one vector $\nabla h(x)$ is linearly dependent if there exists $a \neq 0$ such that $a\nabla h(x) = 0$. However, this is only true if $x = 0$, which is not feasible because $h(0) \neq 0$. Thus, because $\nabla h(x)$ cannot be linearly dependent with feasible points, all feasible points are regular. The tangent plane at a feasible point x^* is

$$\left\{\Delta x \in \mathbb{R}^3 \ : \ x_1^*\Delta x_1 + x_2^*\Delta x_2 + x_3^*\Delta x_3 = 0\right\}$$

where the coefficient of 2 is dropped.

Example 2. For $h_1(x) = x_1^2 + x_2^2 - 1$ and $h_2(x) = x_3 - 1$ we have

$$\nabla h_1(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 0 \end{bmatrix} \qquad \nabla h_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The only points on which these vectors are linearly dependent occurs at $x = 0$, but it is not a feasible point. Thus, all feasible points are regular. The tangent plane at the feasible point x^* is defined by:

$$\left\{\Delta x \in \mathbb{R}^3 \ : \ x_1^*\Delta x_1 + x_2^*\Delta x_2 = 0, \ \Delta x_3 = 0\right\}$$

Consider a local analysis of the objective function around x^*

$$\begin{array}{ll} \min\limits_{x} \ f_0(x) & \\ \text{s.t. } x \in \mathcal{D} & \end{array} \longrightarrow \begin{array}{l} \min\limits_{\Delta x} \ f_0(x^* + \Delta x) \\ \text{s.t. } \Delta x \in \text{tangent plane at } x^* \text{ and } \Delta x \text{ small} \end{array}$$

and so

$$\begin{aligned} f_0(x^* + \Delta x) &= f_0(x^*) + \nabla f_0(x^*)^\top \Delta x + \text{h.o.t.} \\ &\geq f_0(x^*) \ \text{ by local optimality condition} \end{aligned}$$

Theorem 7.4.1 If x^* is regular and a local minimum, then for every Δx such that $\nabla h_i(x^*)^\top \Delta x = 0, \ i = 1, \ldots, p$, the term $\nabla f_0(x^*)^\top \Delta x$ is nonnegative, i.e. $\nabla f_0(x^*)^\top \Delta x \geq 0$.

Since Δx and $-\Delta x$ are both valid perturbations, the constraint from Theorem 7.4.1 implies that $\nabla f_0(x^*)^\top \Delta = 0$[13]. Since $\nabla h_i(x^*)^\top \Delta x = 0$ for $i = 1, \ldots, p$ for every small Δ, it must be the case that $\nabla f_0(x^*)$ must lie in the same subspace spanned by $\nabla h_i(x^*)$ for $i = 1, \ldots, p$. This leads to Theorem 7.4.2.

[13] If $\nabla f_0(x^*)^\top \Delta > 0$, then $\nabla f_0(x^*)^\top(-\Delta) < 0$, which violates condition specified in Theorem 7.4.1.

Theorem 7.4.2 First order necessary optimality condition. Under the conditions of Theorem 7.4.1, $\exists\ \nu_1^*, \ldots, \nu_p^*$ such that

$$\nabla f_0(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0$$

that is, the gradient of f_0 at a local optimum x^* is a linear combination of the gradient of constraints.

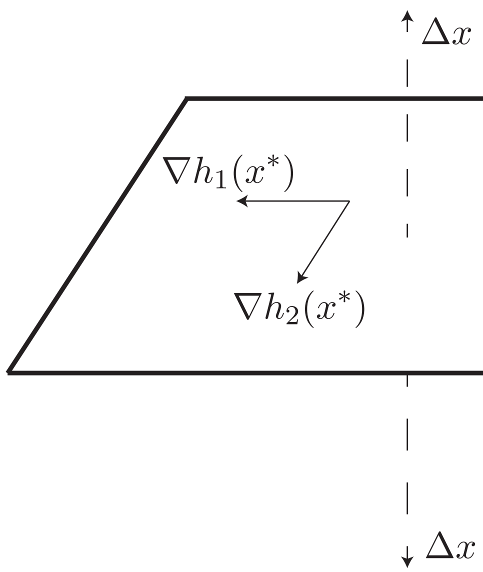

Figure 23: Tangent plane formed by $\nabla h_1(x^*)$ and $\nabla h_2(x^*)$. As can be seen, Δx is orthogonal to the tangent plane.

Figure 23 depicts the idea in Theorem 7.4.2.

From Theorem 7.4.2 we can derive the second order necessary optimality condition for local optimal solution x^*. Starting from the Taylor series expansion

$$\begin{aligned}
f_0(x^* + \Delta x) &= f_0(x^*) + \nabla f_0(x^*)^\top \Delta x + \frac{1}{2} \Delta x^\top \nabla^2 f_0(x^*) \Delta x + \cdots \\
\nu_1^* h_1(x^* + \Delta x) &= \nu_1^* h_1(x^*) + \nu_1^* \nabla h_1(x^*)^\top \Delta x + \frac{\nu_1^*}{2} \Delta x^\top \nabla^2 h_1(x^*) \Delta x + \cdots \\
\vdots \qquad & \qquad\qquad \vdots \\
\nu_p^* h_p(x^* + \Delta x) &= \nu_p^* h_p(x^*) + \nu_p^* \nabla h_p(x^*)^\top \Delta x + \frac{\nu_p^*}{2} \Delta x^\top \nabla^2 h_p(x^*) \Delta x + \cdots
\end{aligned}$$

We are interested in adding all the previous equations. Note that $h_i(x^*)$ and $h_i(x^* + \Delta x)$ are all equal to zero $\forall i = 1, \ldots, p$ when x^* is feasible. As discussed previously, $\nabla f_0(x^*)^\top \Delta x = 0$. By Theorem 7.8, $\sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0$. Thus, adding the previous equations, yields

$$
\begin{aligned}
f_0(x^* + \Delta x) &= f_0(x^*) + \frac{1}{2}\Delta x^\top \nabla^2 f_0(x^*)\Delta x + \frac{1}{2}\sum_{i=1}^{p} \nu_i^* \Delta x^\top \nabla^2 h_i(x^*)\Delta x + \text{h.o.t.} \\
&\geq f_0(x^*) \;\; \text{by local optimality}
\end{aligned}
$$

and so

$$
\Delta x^\top \left(\nabla^2 f_0(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla^2 h_i(x^*) \right) \Delta x \geq 0
$$

> **Theorem 7.4.3 Second order necessary optimality condition.** Under the conditions of Theorem 7.4.2 and Theorem 7.4.1, we have
>
> $$M = \Delta x^\top \left(\nabla^2 f_0(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla^2 h_i(x^*) \right) \Delta x \geq 0$$
>
> for every Δx such that $\nabla h_i(x^*)^\top \Delta x = 0,\ i = 1, \dots, p$ for local minimum solution x^*.

A particular case when there are no constraints:

$$
\Delta x^\top \nabla^2 f_0(x^*)\Delta x \geq 0 \;\Rightarrow\; \nabla^2 f_0(x^*) \succeq 0
$$

If $M = 0$, the difference between $f_0(x^* + \Delta x)$ and $f_0(x^*)$ should be quantified by third order terms or higher.

> **Theorem 7.4.4 Second order sufficient optimality condition.** If x^* is regular and feasible, for which $\exists\ \nu_1^*, \dots, \nu_p^*$, such that first order optimality condition is satisfied and $M > 0$ for every Δx in the tangent plane at the point x^* such that $\Delta x \neq 0$, then x^* is a local minimum.

If the optimization problem is unconstrained, Theorem 7.11 implies that the conditions

$$
\nabla f_0(x^*) = 0 \quad \text{and} \quad \nabla^2 f_0(x^*) \succ 0
$$

are sufficient to establish x^* as a local minimum.

How to check second order conditions?

$$
\text{Tangent plane} = \mathcal{T} = \{\Delta x \;:\; \nabla h_i(x^*)^\top \Delta x = 0,\ i = 1, \dots, p\}
$$

Note that $\dim(\mathcal{T}) = n - p$ because x^* is a regular point. Pick $n - p$ linearly independent vectors in the tangent plane $e_1, e_2, \dots, e_{n-p} \in \mathbb{R}^n$.

Define $E = [e_1, e_2, \dots, e_{n-p}] \in \mathbb{R}^{n\times(n-p)}$. The tangent plane can then be re-expressed as

$$
\mathcal{T} = \{Ey \;:\; y \in \mathbb{R}^{n-p}\}
$$

Then, the second order necessary condition reduces to

$$E^\top \left(\nabla^2 f_0(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla^2 h_i(x^*) \right) E \succeq 0$$

and the second order sufficient condition to

$$E^\top \left(\nabla^2 f_0(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla^2 h_i(x^*) \right) E \succ 0$$

8.5 Generalization to Inequality Constraints

In this section we apply the first order and second order necessary optimality conditions to optimization problems with inequality as well as equality constraints. Consider the general optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f_0(x) \\ \text{s.t.} \quad & f_1(x) \le 0 \\ & h_1(x) = 0 \end{aligned} \tag{20}$$

can be reformulated with only equality constraints as

$$\begin{aligned} \min_{x\in\mathbb{R}^n, z\in\mathbb{R}} \quad & f_0(x) \\ & f_1(x) + z^2 = 0 \leftarrow \lambda_1 \\ & h_1(x) = 0 \leftarrow \nu_1 \end{aligned}$$

where λ_1 and ν_1 are associated Lagrangian multipliers. Define $\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{n+1}$ along with

- $\tilde{f}_0(\tilde{x}) = f_0(x)$
- $\tilde{h}_1(\tilde{x}) = h_1(x)$
- $\tilde{h}_2(\tilde{x}) = f_1(x) + z^2$

If $\bar{x}^*$ is a regular and local minimum, then it is necessary by the first order necessary optimality condition that

$$\nabla \tilde{f}_0(\tilde{x}^*) + \nu_1^* \nabla \tilde{h}_1(\tilde{x}^*) + \nu_2^* \nabla \tilde{h}_2(\tilde{x}^*) = 0$$

where $\lambda_1^* = \nu_2^*$. The above can be re-expressed as

$$\begin{bmatrix}\nabla f_0(x^*)\\ 0\end{bmatrix} + \nu_1^* \begin{bmatrix}\nabla h_1(x^*)\\ 0\end{bmatrix} + \nu_2^* \begin{bmatrix}\nabla f_1(x^*)\\ 2z^*\end{bmatrix} = \mathbf{0} \in \mathbb{R}^{n+1}$$

The only way the last coordinate is 0 is if $2\lambda_1^* z^* = 0$, which implies that $\lambda_1^*(z^*)^2 = 0$[14]. By the required constraint, $f_1(x^*) + z^2 = 0 \Rightarrow \lambda_1(f_1(x^*) + z^2) = 0$. Since $\lambda_1 z^2 = 0$, then $\lambda_1 f_1(x^*) = 0$, which is a complementary slackness condition for local optimality of x^*. From the second order optimality condition

$$(\Delta\tilde{x})^\top \left(\nabla^2 \tilde{f}_0(\tilde{x}^*) + \nu_1^* \nabla^2 \tilde{h}_1(\tilde{x}^*) + \nu_2^* \nabla^2 \tilde{h}_2(\tilde{x}^*)\right) \Delta\tilde{x} \geq 0$$

where

$$\nabla^2 \tilde{f}_0(\tilde{x}^*) = \begin{bmatrix}\nabla^2 f_0(x^*) & 0\\ 0 & 0\end{bmatrix}, \quad \nabla^2 \tilde{h}_1(\tilde{x}^*) = \begin{bmatrix}\nabla^2 h_1(x^*) & 0\\ 0 & 0\end{bmatrix}, \quad \nabla^2 \tilde{h}_2(\tilde{x}^*) = \begin{bmatrix}\nabla^2 f_1(x^*) & 0\\ 0 & 2\end{bmatrix}$$

which leads to

$$\Delta\tilde{x}^\top \begin{bmatrix}\nabla^2 f_0(x^*) + \nu_1^* \nabla^2 h_1(x^*) + \lambda_1^* \nabla^2 f_1(x^*) & 0\\ 0 & 2\lambda_1^*\end{bmatrix} \Delta x^\top \geq 0$$

which implies that $2\lambda_1^* \geq 0$ must be true. If x^* is regular, then $\tilde{x}^*$ is also regular for the reformulated problem. Consider the two cases

1. $z^* = 0$ implies that $f_1(x)$ constraint, and so $\nabla h_1(x^*)$ and $\nabla f_1(x^*)$ are linearly independent from the regularity of x^*, that implies that $\nabla \tilde{h}_1(\tilde{x}^*)$ and $\nabla \tilde{h}_2(\tilde{x}^*)$ are linearly independent.

2. $z^* \neq 0$ implies that $f_1(x)$ is not binding. Then from the regularity of x^* we know that $\nabla h_1(x) \neq 0$. The previous, and that $z^* \neq 0$, implies that $\nabla \tilde{h}_1(\tilde{x}^*)$ and $\nabla \tilde{h}_2(\tilde{x}^*)$ are linearly independent.

Taken together, this leads the first and second order necessary condition for local optimality for optimization problems with equality and inequality constraints.

[14] For $\lambda z = 0$, either $\lambda = 0$ or $z = 0$. If $\lambda = 0$, then $\lambda z^2 = 0$. Similarly if $z = 0$, then $\lambda z^2 = 0$

Theorem 7.5.1 First order necessary condition for optimization problem of the form

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0 \quad i = 1, \ldots, m \\ & h_j(x) = 0 \quad j = 1, \ldots, p \end{aligned}$$

If x^* is regular and a local minimum, then $\exists\ \lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ and $\nu_1^*, \ldots, \nu_p^*$ such that the following conditions are necessary

1. $\lambda_i^* \geq 0 \quad i = 1, \ldots, m$
2. Complementary slackness condition: $\lambda_i^* f_i(x^*) = 0$ for $j = 1, \ldots, p$
3. $\nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{j=1}^p \nu_i^* \nabla h_i(x^*) = 0$

Theorem 7.5.2 For the optimization problem form in Theorem 7.5.1, the second order necessary conditions for regular point x^* to be a local minimum are

1. Conditions 1, 2, and 3 stated in Theorem 7.5.1.
2. $(\Delta x)^\top \left(\nabla^2 f_0(x^*) + \sum_{i=1}^m \nu_i^* \nabla^2 h_i(x^*) + \sum_{j=1}^p \lambda_j^* \nabla^2 f_j(x^*)\right) \Delta x \geq 0$

For every Δx in the tangent plane at x^*

Theorem 7.5.3 Second order sufficient condition for local optimality of x^* for optimization problem of equality and inequality constraints. If x^* is feasible and a regular point for which $\exists\ \lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ and $\nu_1^*, \ldots, \nu_p^*$ such that conditions 1, 2, and 3 from Theorem 7.5.2 are satisfied, and condition 4 from Theorem 7.5.2 is satisfied in a strict way whenever $\Delta x \neq 0$ and $\Delta x \in \mathcal{T}$, then these conditions are sufficient for x^* to be a local minimum. Where

$$\begin{aligned} \mathcal{T} = \{\Delta x \mid \nabla h_i(x^*)^\top \Delta x = 0 \quad i = 1, \ldots, p; \\ \nabla f_i(x^*) \Delta x = 0,\ \text{if } f_i(x^*) = 0 \text{ and } \lambda_i > 0\} \end{aligned}$$

The second order sufficient condition guarantees strict local optimality of x^*.

In $\mathcal{T}$ we are considering only the active constraints that are non degenerate (which satisfies $\lambda > 0$). $\mathcal{T}$ is bigger than the tangent plane, since it has less constraints.

For an optimization problem with inequality and equality constraints as in Equation 20 but with m inequality constraints and p equality constraints, a inequality constraint $f_i(x^*) \leq 0$ is active if $f_i(x^*) = 0$ and $\lambda_i^* \geq 0$. If these conditions are true, then $\nabla f_i(x^*)^\top \Delta x = 0$, for $\Delta x \in \mathcal{T}$ and $i = 0$ and active inequality constraints.

Below are definitions of a regular point x^*.

- Point x^* is regular if gradients of equality and all active inequality constraints are linearly independent.

- Tangent plane at x^* is the set of all $\Delta x \in \mathbb{R}^n$ that are orthogonal to gradients of equality and active inequality constraints.

8.6 Sensitivity Analysis

Consider the perturbed version of a general optimization problem in Equation 17

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq c_i, i = 1, \ldots, m \\ & h_j(x) = d_j, j = 1, \ldots, p \end{aligned}$$

Define $p^*(c, d)$ as the optimal value of the perturbed problem. Then $p^* = p^*(0, 0)$. If the original problem in Equation 17 is convex then $p^*(c, d)$ is convex[15].

Example 1
Consider the original convex optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & x \\ \text{s.t.} \quad & x^2 \leq 0 \end{aligned}$$

which has solution $x^* = 0$ and optimal value $p^*(0, 0) = 0$. If perturbation is introduced such that the constraint is replaced with $x^2 \leq c$, then $-\sqrt{c} \leq x \leq \sqrt{c}$. Then the solution is

$$p^*(c) = \begin{cases} -\sqrt{c} & c \geq 0 \\ \text{infeasible} & c < 0 \end{cases}$$

which is a convex function in c.

Assume second order sufficient condition is satisifed for x^* and no constraint is degenerate[16]. Then there exists a ball centered around $(0, 0)$ such that for every $(c, d) \in$ ball, we have

1. $p^*(c, d)$ exists.
2. There is a solution $x^*(c, d)$ such that $x^*(0, 0) = x^*$ and continuous[17].
3. $\nabla_c p^*(c, d)\Big|_{(0,0)} = -\lambda^*$
4. $\nabla_d p^*(c, d)\Big|_{(0,0)} = -\nu^*$

[15] The proof is given on http://www.ifp.illinois.edu/~angelia/L10_sensitivity.pdf

[16] A degenerate inequality constraint is an active constraint $f_i(x^*) = 0$ with dual variable $\lambda_i^* = 0$.

[17] By continuous, meaning $\lim_{(c,d) \to (0,0)} x^*(c, d) = x^*(0, 0)$

For small perturbations (c, d)

$$
\begin{aligned}
p^*(c,d) &= \min_x \max_{\lambda \geq 0, \nu} f_0(x) + \sum_{i=1}^m \lambda_i (f_i - c_i) + \sum_{j=1}^p \nu_j (h_j(x) - d_j) \\
&= \min_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{j=1}^p \nu_j^* h_j(x) \right) - \sum_{i=1}^m \lambda_i^* c_i - \sum_{j=1}^p \nu_j^* d_j \\
&= p^* - \sum_{i=1}^m \lambda_i^* c_i - \sum_{j=1}^p \nu_j^* d_j
\end{aligned}
$$

Properties (3) and (4) above following naturally as a consequence. If λ_i^* and ν_j^* are small for all i and j, then the perturbation doesn't affect the solution $p^*(c, d)$ from p^* much. The second order necessary condition stated in Theorem 7.5.2 involving Hessians is automatically satisfied for a convex optimization problem because $\nabla^2 f_0(x^*) \succeq 0$, $\nabla^2 f_i(x^*) \succeq 0$ for all i, and $\nabla^2 h_j(x^*) = 0$ for all j since $h_j(x)'s$ are linear.

The second order sufficient condition according to Theorem 7.5.3 requires strict $\succ 0$, but this strict condition is not required for convex optimization problem. For example, since a LP only involves linear functions, all terms $\nabla^2 f_0(x^*), \nabla^2 f_i(x^*), \nabla^2 h_j(x^*) = 0$.

In summary:

1. If x^* is regular and a local minimum, then first order optimality condition is satisfied.
2. If x^* is regular, feasible, and satisfies first order optimality condition, then it is a global minimum for convex optimization problems.
3. First order optimality condition is necessary and sufficient under regularity conditions for convex optimization problems.

8.7 Slater's Condition

Consider a convex optimization problem of the form in Equation 17. If a convex optimization problem satisfies the KKT conditions, then $\mathcal{L}(x^*, \lambda^*, \nu^*) = f_0(x^*)$, due to complementary slackness of primal feasibility. The stationarity condition of KKT and $\mathcal{L}(x^*, \lambda^*, \nu^*) = f_0(x^*)$ both imply that $\min_x \mathcal{L}(x, \lambda^*, \nu^*) = p^*$, since first order conditions are sufficient for convex problems under regularity conditions. This in turn implies that (λ^*, ν^*) are geometric multipliers.

For convex optimization problems, the regularity conditions on x^* for establishing strong duality can be replaced with Slater's condition.

Definition 1: Slater's condition is satisfied if $\exists \bar{x} \in \mathbb{R}^n$ that is feasible and $f_i(\bar{x}) < 0$ for all i. Note $\bar{x}$ is an arbitrary point, not necessarily optimal.

Definition 2: Weak form of Slater's condition only requires feasible point to satisfy all nonlinear inequalities in a strict way. In other words, strict condition does not need for affine inequality constraints.

Example 1
Consider the following constraints

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 &\leq 0 \\ x_1^2 + x_3^2 &\leq 0 \end{aligned}$$

Then Slater's condition requires strict inequality for the last two constraints while the weak form of Slater's condition requires strict inequality only for the last constraint.

Theorem 7.7.1

1. Weak duality always holds for convex and nonconvex problems.
2. Strong duality holds for convex optimization problems under weaker form of Slater's condition.
3. Under (2), if the optimal objective value is finite, then there is a dual solution.

Example 2

- $\min e^x \;\rightarrow\; x^* = -\infty$, no solution.
- $\min -e^x \;\rightarrow$ optimal value is not finite.
- $\inf -1/x \;\rightarrow\; x \rightarrow 0^+ \;\rightarrow\; -1/x \rightarrow -\infty$ optimal value is not finite, but a solution can be expressed as $x \rightarrow 0^+$.

Let (p) refers to the primal problem and (d) to the dual problem. If Slater is satisfied, $p^* = d^*$, and implies

- If p^* is finite and Slater's condition is satisfied for primal problem, then $\exists$ finite (λ^*, ν^*).
- If d^* is finite and Slater's condition is satisfied for dual problem, then $\exists$ finite x^*.

Example 3: LP problems
Consider (p) a LP primal and its associated dual (d). The (p) of a LP is

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \;\; & c^\top x \\ \text{s.t.} \;\; & Ax = b \\ & x \succeq 0 \end{aligned}$$

From Example 1 of section 7, the dual problem of (p) is another LP

$$\begin{aligned} \min_{\nu,\lambda} \quad & -b^\top \nu \\ & c + A^\top \nu - \lambda = 0 \\ & \lambda \succeq 0 \end{aligned}$$

Which can be further simplified to

$$\begin{aligned} \min_{\nu} \quad & -b^\top \nu \\ & c + A^\top \nu \succeq 0 \end{aligned}$$

since if $c + A^\top \nu \succeq 0$ is true, then there exists $\lambda \succeq 0$ such that $c + A^\top \nu = \lambda$.

We can use weaker Slater if $\exists \bar{x} : A\bar{x} = b, \bar{x} \geq 0$ (i.e. feasibility). In that case if (p) if feasible (or (d) is feasible) then $p^* = d^*$. In general we have the following cases:

$$p^* = \begin{cases} +\infty & : \text{ infeasible} \\ \text{finite} & : \text{ good} \\ -\infty & : \text{ unbounded from below} \end{cases} \quad \wedge \quad d^* = \begin{cases} +\infty & : \text{ unbounded from above} \\ \text{finite} & : \text{ good} \\ -\infty & : \text{ infeasible} \end{cases}$$

The 3 scenarios are:

i. $p^* = d^* = +\infty \Rightarrow$ (p) is infeasible.

ii. $p^* = d^* = -\infty \Rightarrow$ (d) is infeasible.

iii. $p^* = d^* = \text{finite} \Rightarrow$ both (p) and (d) have solutions.

The only way that strong duality does not hold is if both problems are infeasible:

$$p^* = \infty \text{ and } d^* = -\infty \Rightarrow \text{gap} = p^* - d^* = +\infty$$

Theorem 7.7.2
A LP problem has a zero duality gap unless (p) and (d) are both infeasible.

Example 4
Consider the following non-convex optimization problem[18]

$$\begin{aligned} \min_{x \in \mathbb{R}^n} (\text{or} \max_{x \in \mathbb{R}^n}) \quad & \sum_{i \neq j} x_i x_j \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i^2 = n \end{aligned}$$

This problem is easy to solve because of the equality

[18] The problem is non-convex because the feasible set is not convex.

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i\neq j} x_i x_j \ \rightarrow \ \sum_{i\neq j} x_i x_j = \frac{1}{2}\left(\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2\right) = \frac{1}{2}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{n}{2}$$

The maximum, from Cauchy-Schwartz, is attained when $\mathbf{1}$ is colinear with x (that gives solution $x = \mathbf{1}$ and $x = -\mathbf{1}$), while the minimum is attained when $\mathbf{1}^\top x = 0$. For the purpose of this problem we ignore the solution and write the Lagrangian as:

$$\mathcal{L}(x,\nu) = \sum_{i\neq j} x_i x_j + \nu\left(\sum_{i=1}^{n} x_i^2 - n\right)$$

KKT conditions yield:

1) $\nabla_x \mathcal{L}(x,\nu) = \sum_{i\neq j} x_j + 2\nu x_i = \sum_{j=1}^{n} x_j + (2\nu - 1)x_i = 0,\ i = 1,\ldots,n.$

2) $\sum_{i=1}^{n} x_i^2 = n.$

Where for the last equality of 1), x_i is added to the first term and x_i is subtracted from the second term. Note that since there are no inequality constraints, additional conditions such as complementary slackness do not apply (Theorem 7.5.1).

Depending on what ν is, there are two ways to satisfy 1):

i. $2\nu - 1 = 0 \rightarrow \nu = \frac{1}{2} \rightarrow \sum_{i=1}^{n} x_i = 0.$

ii. $2\nu - 1 \neq 0 \rightarrow x_i = \frac{-\sum_{j=1}^{n} x_j}{2\nu - 1},\ i = 1,\ldots,n.$ This implies $x_1 = x_2 = \ldots = x_n$ for a locally optimal solution[19]

From the result of ii.), the equality constraint can be rewritten as $nx_i^2 = n$, which implies that $x_i^2 = 1$. To satisfy $x_i = (-\sum_{j=1}^{n} x_j)/(2\nu - 1)$ from ii) whether $x_i = 1$ or $x_i = -1$

a) If $x_i = 1$ for every i, then

$$1 = \frac{-n}{2\nu - 1} \Rightarrow \nu = -\frac{n-1}{2}$$

b) If $x_i = -1$ for every i, then

$$-1 = \frac{n}{2\nu - 1} \Rightarrow \nu = -\frac{n-1}{2}$$

[19] To see why $x_i = x_j$ for all $i \neq j$, notice that $2\nu - 1 = -\sum_{j=1}^{n} x_j/x_i$ for all i. Regardless of whether $x_i = x_j$ for all $i \neq j$, the numerator on the right hand side will be the same for all i. However, if $x_i \neq x_j$ for some $i \neq j$, then the equality cannot be true for both i and j.

For checking the second order optimality condition, we first check whether the regularity condition that $\nabla h_1(x), \dots \nabla h_p(x)$ are linearly independent for all feasible x. In this example, this means that:

$$\alpha \nabla h_1(x) = \alpha \begin{bmatrix} 2x_1 \\ \vdots \\ 2x_n \end{bmatrix} = 0$$

for some $\alpha \neq 0$. This is only true for $x = 0$, but $x = 0$ is not feasible. Thus, all feasible points are regular.

Now we construct the tangent plane at $x^* = \mathbf{1}_n$ as:

$$T = \{\Delta x \mid \nabla h_i(x^*)^\top \Delta x \to \sum_{i=1}^{n} \Delta x_i = 0\}$$

Recall that the tangent plane could be alternatively rewritten as

$$\mathcal{T} = \{Ey \ : \ y \in \mathbb{R}^{n-p}\}$$

where $E \in \mathbb{R}^{n \times n-p}$ is a matrix whose columns are linearly independent and p is the number of equality constraints.

To find E in this example we need to find $n-1$ linearly independent vectors. Using

$$e_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \; \dots \; e_{n-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$$

we can define $E = [e_1, e_2, \dots, e_{n-1}]$.

Then, the second order condition can be written as

$$\begin{aligned} E^\top (\nabla^2 f_0(x^*) + \nu^* \nabla^2 h_i(x^*)) E &= E^\top \left(\begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} - \frac{n-1}{2} \mathrm{diag}(2) \right) E \\ &= E^\top (\mathbf{1}_n \mathbf{1}_n^\top - I - (n-1)I) E \end{aligned}$$

where $\nu^* = -(n-1)/2$ as derived previously. Note that $\mathbf{1}_n^\top E = 0$ and so:

$$
\begin{aligned}
E^\top(\nabla^2 f_0(x^*) + \nu^*\nabla^2 h_i(x^*))E &= -E^\top(-nI)E \\
&= -nE^\top E \\
&= -n\begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 2 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 2 \end{bmatrix} \\
&= -n(\mathbf{1}_n\mathbf{1}_n^\top + I) \\
&\prec 0
\end{aligned}
$$

Then, the second order condition is sufficient for local maximization. The same E works for $x^* = -\mathbf{1}$ (since they yield the same tangent plane $\sum_{i=1}^n \Delta x_i = 0$). Thus, for $x^* = -\mathbf{1}_n$

$$E^\top(\nabla^2 f_0(x^*) + \nu^*\nabla^2 h_i(x^*))E \prec 0$$

which implies the second order sufficient optimality condition is met if the optimization problem was a maximization instead of a minimization problem.

Considering the case when $2\nu - 1 = 0$

$$\sum_{i=1}^n x_i^2 = n \;\wedge\; \sum_{i=1}^n x_i = 0$$

In $\mathbb{R}^2$ ($n = 2$) this is the intersection of the line $x_2 = -x_1$ and the circle $x_1^2 + x_2^2 = 2$. This yields two isolated points given by $x^{(1)} = (1, -1)$ and $x^{(2)} = (-1, 1)$ that achieves local minima.

In the case of $\mathbb{R}^n$ with $n \geq 3$, the two constraints are the intersection of a hyperplane with a n-sphere, that generate points that are not isolated. Since we have first order optimality condition for non isolated points, the second order sufficient conditions cannot be satisfied for $n \geq 3$[20]. However, it can be checked that the second order necessary condition for local minima is satisfied for these points. Moreover, we know that

$$\sum_{i\neq j} x_i x_j = \underbrace{\frac{1}{2}\left(\sum_{i=1}^n x_i\right)^2}_{\geq 0} - \underbrace{\frac{1}{2}\left(\sum_{i=1}^n x_i^2\right)}_{=n} \geq 0 - \frac{n}{2}$$

on which the equality holds when $\sum_{i=1}^n x_i = 0$ is attained. Thus, these points given by the intersection of the hyperplane and n-sphere are global minima with optimal value $-n/2$.

In addition, the previously found points $\mathbf{1}_n$ and $-\mathbf{1}_n$ that are local maxima are actually global maxima, since

1. We know that all feasible points are regular and we have analyzed every possible stationary

[20] The reason for this is not explicitly discussed in class, but a potential explanation for this is that if the second order sufficient conditions are satisfied at a given point, then the neighboring points will yield a higher objective value. However, this implication is not compatible with the first order conditions of local optimality.

point from the KKT conditions, that are necessary for local optimality.

2. The feasible set is a compact set (it is closed since it contains its boundary and it is bounded since it its contained in the ball of radius n). Under these conditions a global solution exists (see Theorem 7.7.3)

3. Both $\mathbf{1}_n$ and $-\mathbf{1}_n$ give the same optimal objective value $n(n-1)/2$. Since this value is greater than any other stationary point, these points are actually global maxima.

Theorem 7.7.3 Extreme Value Theorem If $f_0(x)$ is continuous and the feasible set of the optimization problem is compact, then there exists global minimum and maximum.

Proof: See Theorem 3.4 in
`http://www.math.uchicago.edu/~may/VIGRE/VIGRE2008/REUPapers/Murphy.pdf`

Example 5:
Consider the following QP optimization problem:

$$\begin{aligned} &\min_x \frac{1}{2}x^\top Px + q^\top x + r \\ &\text{s.t. } Ax = b \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}$ and $P \in \mathbb{S}_+^n$. If the problem is feasible, Weak Slater holds for QP and optimality condition is equivalent to KKT:

- Primal feasibility: $Ax = b$
- Stationarity:

$$\begin{aligned} 0 &= \nabla_x \mathcal{L}\left(\frac{1}{2}x^\top Px + q^\top x + r + \nu^{*\top}(Ax - b)\right) \\ &= Px + q + A^\top \nu^* \end{aligned}$$

Stationarity and primal feasibility can be compactly written as:

$$\underbrace{\begin{bmatrix} P & A^\top \\ A & 0 \end{bmatrix}}_{M} \begin{bmatrix} x \\ \nu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

There are 3 possibilities: i) no solution, ii) unique solution or iii) infinitely many solutions. If M is invertible, then there exists a unique solution given by $(x^{*\top}, \nu^{*\top})^\top = M^{-1}(-q^\top, b^\top)^\top$. Otherwise, there are no solutions or infinitely many solutions.

Example 6:
Consider the following problem

$$\begin{aligned} &\min_{x_1,x_2} \; x_2 \\ &\text{s.t. } x_1 = 0 \end{aligned}$$

which is unbounded below for $x_2 = -\infty$.

Example 7:
Consider the following QCQP problem

$$\begin{aligned} &\min \ \frac{1}{2}x^\top P_0 x + q_0^\top x + r_0 \\ &\text{s.t.} \ \frac{1}{2}x^\top P_i x + q_i^\top x + r_i \leq 0, \ i = 1, \ldots, m \end{aligned} \tag{21}$$

If any P_i is not PSD, then the problem is non-convex, and non-zero duality gap in general does not hold.

> **Theorem 7.7.4 $\mathcal{S}$-procedure** Consider a QCQP problem defined in (21). If $m = 1$ and Slater holds ($\exists \bar{x}$ in the interior) then duality gap is zero, even for a non convex problem.

8.8 Theorem of Alternatives for Non-linear Case

Consider a (possibly non-convex) feasible set

$$\mathcal{X} = \{x \mid f_i(x) \leq 0, \ i = 1, \ldots, m, h_j(x) = 0, \ j = 1, \ldots, p\}$$

We are interested in checking feasibility or infeasibility of the set. Consider the following optimization problem

$$\begin{aligned} p^* = &\min_x \ 0 \\ &\text{s.t.} \ f_i(x) \leq 0, \ i = 1, \ldots, m \\ &\qquad h_j(x) = 0, \ j = 1, \ldots, p \end{aligned} \tag{22}$$

If $p^* = 0$, then the problem is feasible, and if $p^* = +\infty$ our problem is infeasible.

Consider its dual function

$$g(\lambda, \nu) = \inf_x \ \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)$$

and the dual problem

$$d^* = \max_{\lambda \geq 0, \nu} g(\lambda, \nu)$$

If $g(\lambda, \nu)$ is strictly positive for some (λ, ν), then due to the linearity of g

$$g(\alpha\lambda, \alpha\nu) = \alpha g(\lambda, \nu) \ \to +\infty \text{ as } \alpha \to +\infty$$

Then

$$d^* = \begin{cases} +\infty & \text{if } \exists(\lambda,\nu) \text{ where } g(\lambda,\nu) > 0 \text{ s.t. } \lambda \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We refer as condition $(*)$ if $\exists(\lambda,\nu)$ where $g(\lambda,\nu) > 0$ s.t. $\lambda \geq 0$. By weak duality we know that $p^* \geq d^*$

i. If $d^* = +\infty$ then $p^* = +\infty$. This implies that if $(*)$ is feasible then the primal problem (22) is infeasible.

ii. If $p^* = 0$ then $d^* = 0$. This implies that if the primal problem (22) feasible then $(*)$ is infeasible.

iii. Cannot conclude anything about whether condition $(*)$ can be met if (22) is infeasible ($p^* = \infty$) since d^* could be zero or $+\infty$. If $(*)$ is infeasible ($d^* = 0$) nothing can be concluded about (22), since p^* could be zero or $+\infty$. Both problems could be infeasible, but cannot be feasible at the same time).

This is referred as **weak alternatives**, on which (22) and $(*)$ cannot be both feasible at the same time.

Theorem 7.8.1 Theorem of Strong Alternatives Consider (22) to be convex. Equality constraints must be affine and since we want Slater to hold, we focus on $f_i(x) < 0$. Define

$$(p) : \begin{cases} f_i(x) < 0, \ i = 1, \dots, m \\ Ax = b \end{cases} \qquad (*) : \begin{cases} \lambda \geq 0 \\ \nu \neq 0 \\ g(\lambda,\nu) \geq 0 \end{cases}$$

Then (p) is feasible if and only if $(*)$ is infeasible.

8.9 Duality Examples

8.9.1 Conic Duality

Consider the following linear conic program

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & a^\top x \\ \text{s.t. } & Ax = b \\ & A_i x - b_i \preceq_{\mathcal{K}_i} 0, \ i = 1, \dots, m \end{aligned}$$

where $A_i x - b_i \preceq_{\mathcal{K}_i} 0$ represents that $-(A_i x - b_i) \in \mathcal{K}_i$, where $\mathcal{K}_i$ is a proper cone.

To find its dual, let $\nu \in \mathbb{R}^m$ be the dual variable associated to the constraint $Ax - b = 0$ and $\lambda_i \in \mathbb{R}^{n_i}$ with $\lambda_i \succeq_{\mathcal{K}_i^*} 0$ be the associated dual variable to the constraint $A_i x - b_i \preceq_{\mathcal{K}_i} 0$. The dual variable λ_i is correctly constrained to belong to the dual cone $\mathcal{K}_i^*$ because by definition of the dual cone

$$\max_{\lambda \in \mathcal{K}^*} -\lambda^\top (Ax-b) = \begin{cases} 0 & \text{if } Ax-b \in \mathcal{K} \\ \infty & \text{otherwise} \end{cases}$$

If $Ax - b \in \mathcal{K}$, then $-\lambda^\top (Ax-b) \leq 0$, so choosing $\lambda = 0$ will yield optimal value 0. If $Ax - b \notin \mathcal{K}$, then $\lambda^\top (Ax-b) < 0$ by definition, so the optimal objective value becomes ∞. The above outcomes allows proper construction of the Lagrangian function because if the constraint $A_i x - b_i \in \mathcal{K}$ is violated, then the primal problem has objective value $p^* = \infty$ and is thus infeasible.

The dual function $g(\lambda, \nu)$ is defined as

$$\begin{aligned} g(\lambda,\nu) &= \inf_x \mathcal{L}(x,\lambda,\nu) \\ &= \inf_x f_0(x) + \sum_{i=1}^m \lambda_i^\top f_i(x) + \sum_{j=1}^p \nu_i h_i(x) \\ &= \inf_x a^\top x + \sum_{i=1}^m \lambda_i^\top (b_i - A_i x) + \nu^\top (Ax - b) \\ &= \inf_x \sum_{i=1}^m \lambda_i^\top b_i - \nu^\top b + \left(a - \sum_{i=1}^m A_i^\top \lambda_i + A^\top \nu \right)^\top x \\ &= \begin{cases} \sum_{i=1}^m \lambda_i^\top b_i - \nu^\top b & \text{if } a - \sum_{i=1}^m A_i^\top \lambda_i + A^\top \nu = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Thus the dual problem becomes

$$\begin{aligned} \max_{\lambda,\nu} \quad & \left(\sum_{i=1}^m \lambda_i^\top b_i \right) - \nu^\top b \\ & a - \sum_{i=1}^m A_i^\top \lambda_i + A^\top \nu = 0 \\ & \lambda_i \succeq_{K_i} 0, \ i = 1, \dots, m \end{aligned}$$

8.9.2 SDP Duality

Recall SDP in standard form

$$\begin{aligned} \min_{X \in \mathbb{R}^{n\times n}} \quad & \operatorname{tr}(M_0 X) \\ \text{s.t.} \quad & \operatorname{tr}(M_i X) = a_i \quad i = 1, \dots, m \\ & X \succeq 0 \end{aligned}$$

where $M_0, M_1, \dots, M_m \in \mathbb{S}^n$. We define the following dual variables for the constraints

- $\text{Tr}(M_i X) = a_i \leftarrow \nu_i \quad i = 1, \dots, m$
- $-X \preceq 0 \leftarrow W \succeq 0$

The dual variable $W \succeq 0$ is correct because the dual cone of PSD matrices is itself. Using a similar argument used in 8.9.1 conic duality

$$\max_{W \geq 0} -\text{Tr}(WX) = \begin{cases} 0 & \text{if } X \succeq 0 \\ \infty & \text{otherwise} \end{cases}$$

which yields an infeasible primal objective value if the constraint $X \succeq 0$ is not met. The Lagrangian function is then defined as

$$\begin{aligned} \mathcal{L}(X, \nu, W) &= \text{Tr}(M_0 X) + \sum_{i=1}^{m} \nu_i (\text{Tr}(M_i X) - a_i) + \text{Tr}((-X)W) \\ &= \text{Tr}((M_0 + \sum_{i=1}^{m} \nu_i M_i - W)X) - \sum_{i=1}^{m} \nu_i a_i \end{aligned}$$

The dual function is

$$\begin{aligned} g(\nu, W) &= \min_X \mathcal{L}(X, \nu, W) \\ &= \begin{cases} -\sum_{i=1}^{m} \nu_i a_i & \text{if } M_0 + \sum_{i=1}^{m} \nu_i M_i - W = 0 \\ -\infty & \text{Otherwise} \end{cases} \end{aligned}$$

Thus, the dual problem is

$$\begin{aligned} \max_{\nu} \quad & -\nu^\top a \\ \text{s.t.} \quad & M_0 + \sum_{i=1}^{m} \nu_i M_i \succeq 0 \end{aligned}$$

To arrive at the above constraint, notice that this is equivalent to $M_0 + \sum_{i=1}^{m} \nu_i M_i = W$ with $W \succeq 0$. To see why, if $M_0 + \sum_{i=1}^{m} \nu_i M_i \succeq 0$, then there must exist $W \succeq 0$ such that $M_0 + \sum_{i=1}^{m} \nu_i M_i = W$. Thus, it is sufficient to solve the above dual problem, then set $W = M_0 + \sum_{i=1}^{m} \nu_i M_i$ to complete solving for all dual variables. The above problem is itself a SDP problem.

A SDP can be expressed as a linear conic program by vectorizing the matrix. Recall that a vectorization of a matrix $A \in \mathbb{R}^{m \times n}$ produces a vector in $\mathbb{R}^{mn}$ given by:

$$\text{vec}(A) = [a_{1,1}, \dots, a_{m,1}, a_{1,2}, \dots, a_{m,2}, \dots, a_{1,n}, \dots a_{m,n}]$$

With that, a standard SDP problem can be written as the following linear conic program

$$
\begin{aligned}
\min_{X \in \mathbb{S}^n} \quad & \text{vec}(M_0)^\top \text{vec}(X) \\
\text{s.t.} \quad & \text{vec}(M_i)^\top \text{vec}(X) = a_i, \quad i = 1, \ldots, m \\
& X \in \mathcal{K}_{psd}
\end{aligned}
$$

We can again show that the dual problem of a SDP is another SDP by starting with another SDP formulation

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & a^\top x \\
\text{s.t.} \quad & F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0
\end{aligned}
$$

Let $Z \succeq 0$ be the dual variable associated with the constraint, since PSD matrices are self-dual. Thus, the Lagrangian function is

$$
\mathcal{L}(x, Z) = a^\top x + \text{Tr}((F_0 + F_1 x_1 + \cdots + F_n x_n) Z)
$$

The dual function is

$$
\begin{aligned}
g(Z) &= \min_x \mathcal{L}(x, Z) \\
&= \min_x \left(\sum_{i=1}^{n} ((a_i + \text{Tr}(F_i Z)) x_i) + \text{Tr}(F_0 Z) \right) \\
&= \begin{cases} \text{Tr}(F_0 Z) & \text{if } a_i + \text{Tr}(F_i Z) = 0 \ \forall i \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

The dual problem is therefore

$$
\begin{aligned}
\max_{Z} \quad & \text{Tr}(F_0 Z) \\
\text{s.t.} \quad & a_i + \text{Tr}(F_i Z) = 0 \ \forall i \\
& Z \succeq 0
\end{aligned}
$$

which is a SDP in canonical form.

8.9.3 SOCP Duality

Recall the SOCP without equality constraints in standard form

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^T x \\
&||A_i x + b_i||_2 \leq c_i^T x + d_i \quad i = 1, \ldots, m
\end{aligned}
$$

where the problem parameters are $c \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$, $c_i \in \mathbb{R}^n$, and $d_i \in \mathbb{R}$. We can use results of linear conic duality to solve for the dual of SOCP. Recall that a second order cone $\mathcal{K}$ is defined as

$$\mathcal{K} = \{(u, v) \mid u \in \mathbb{R}^n, v \in \mathbb{R}, ||u||_2 \leq v\}$$

Then the SOCP can be expressed as the following linear conic program

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ -\begin{bmatrix} A_i x + b_i \\ c_i^\top x + d_i \end{bmatrix} & \preceq_{\mathcal{K}} 0 \;\; i = 1, \ldots m \end{aligned}$$

To find the dual variable $\begin{bmatrix} u_i \\ v_i \end{bmatrix}$ for $i = 1, \ldots, m$. Recall that since the second order cone is self dual, that this implies $||u_i||_2 \leq v_i$ for all i. In other words, that the dual variable must be the dual cone of a second order cone in order to define a proper Lagrangian function. The dual cone of a second order cone is the set of second order cones, as proved previously.

$$\begin{aligned} \mathcal{L}(x, u, \nu) &= c^\top x - \sum_{i=1}^m \begin{bmatrix} u_i^\top & v_i \end{bmatrix} \begin{bmatrix} A_i x + b_i \\ c_i^\top x + d_i \end{bmatrix} \\ &= \left(c^\top - \sum_{i=1}^m u_i^\top A_i - v_i c_i^\top \right) x - \sum_{i=1}^m u_i^\top b_i - v_i d_i \end{aligned}$$

The dual function is

$$\begin{aligned} g(u, v) &= \min_x \mathcal{L}(x, u, v) \\ &= \begin{cases} -\sum_{i=1}^m u_i^\top b_i - \sum_{i=1}^m v_i d_i & \text{if } c^\top - \sum_{i=1}^m u_i^\top A_i - \sum_{i=1}^m v_i c_i^\top = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

The dual problem is therefore

$$\begin{aligned} \max_{u,v} \quad & \left(-\sum_{i=1}^m u_i^\top b_i - \sum_{i=1}^m v_i d_i \right) \\ \text{s.t.} \quad & c^\top + \sum_{i=1}^m u_i^\top A_i - \sum_{i=1}^m v_i c_i^\top = 0 \\ & ||u_i||_2 \leq v_i \quad i = 1, \ldots m \end{aligned}$$

which is another SOCP problem.

8.10 KKT Condition Examples

8.10.1 Linear Conic Program

Recall a linear conic program in general form

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \preceq_{\mathcal{K}_i} 0 \quad i=1,\dots m \\ & h_i(x) = 0 \quad i=1,\dots,p \end{aligned}$$

Where f_i's and h_i's are linear and $f_1,\dots,f_m$ are vectors. The KKT conditions for some primal-dual triplet (x,λ,ν) are

1. Primal feasibility: $f_i(x^*) \preceq_{\mathcal{K}_i} 0$ for $i=1,\dots,m$; $h_i(x^*)=0$ for $i=1,\dots,p$
2. Dual feasibility: $\lambda_i^* \succeq_{\mathcal{K}_i} 0$ $i=1,\dots,m$
3. Complementary Slackness: $\lambda_i^{*\top} f_i(x^*) = 0$ for $i=1,\dots,m$
4. Stationarity Condition: $\nabla_x \mathcal{L}(x^*,\lambda^*,\nu^*) = 0$

8.10.2 SDP

Recall SDP in standard form

$$\begin{aligned} \min_{X\in\mathbb{R}^{n\times n}} \quad & \mathrm{tr}(M_0X) \\ \text{s.t.} \quad & \mathrm{Tr}(M_iX) = a_i \quad i=1,\dots,m \\ & X \succeq 0 \end{aligned}$$

where $M_0, M_1, \dots, M_m \in \mathbb{S}^n$. The corresponding KKT conditions for the primal-dual triplet (X^*, W^*, ν^*) are[21]

1. Primal feasibility: $\mathrm{Tr}(M_iX^*) = a_i$ for $i=1,\dots,m$ and $X^* \succeq 0$
2. Dual feasibility: $W^* \succeq 0$
3. Complementary Slackness: $\mathrm{Tr}(-X^*W^*) = 0$
4. Stationarity Condition: $\nabla_x \mathcal{L}(x^*,\lambda^*,\nu^*) = 0 \Rightarrow M_0 + \sum_{i=1}^m \nu_i^* M_i - W^* = 0$

[21] Here ν is the dual variable associated with the constraint $\mathrm{Tr}(M_iX) = a_i$ and W is the dual variable associated with the constraint $X \succeq 0$.

8.11 Strong Duality in Linear Conic Programs

If $\exists x \in$ relative interior of the domain of an optimization problem such that conic inequalities are satisfied strictly (i.e. $f_i(x) \prec_{\mathcal{K}_i} 0$), then Slater's condition is satisfied.

Theorem 8.11.1. If Slater's condition is satisfied, then strong duality holds for a linear conic program and the dual optimization problem has a solution. In other words, when Slater's condition is satisfied for linear conic programs, then the KKT conditions also establish global optimality for the primal and dual variables.

Example 1
The Slater's condition for a SDP is that there exists $\bar{X}$ such that $\mathrm{Tr}(M_i \bar{X}) = a_i$ for $i = 1, \dots, m$ and $\bar{X} \succ 0$.

8.12 Fenchel Duality

A conic optimization problem can, in general, be stated as

$$\begin{aligned} \min_x \quad & f(x) \\ & x \in \mathcal{X}_1 \cap \mathcal{X}_2 \end{aligned}$$

where f is a convex function, $\mathcal{X}_1$ is a convex set, and $\mathcal{X}_2$ is a proper cone. To get the intuition for this, consider the linear conic program below

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \; & a_0^\top x \\ \text{s.t.} \; & Ax = b \\ & A_i x - b_i \preceq_{\mathcal{K}_i} 0, \; i = 1, \dots, m \end{aligned}$$

The conic constraint can equivalently be expressed as $\tilde{A}x - \tilde{b} \preceq_{\mathcal{K}} 0$, where

$$\tilde{A} = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_m$$

Thus, for conic optimization, $f(x) = a_0^\top x$ and $\mathcal{X}_1$ is a linear set and $\mathcal{X}_2$ is a proper cone.

Theorem 8.11.1 (Fenchel Duality). Under the assumptions of convex function f_1, concave function f_2, and existence of point in the relative interior of $\mathcal{X}_1 \cap \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are domains of f_1 and f_2 respectively, strong duality holds such that

$$\inf_{x \in \mathcal{X}_1 \cap \mathcal{X}_2} f_1(x) - f_2(x) = \sup_{\nu \in \Lambda_1 \cap \Lambda_2} g_2(\nu) - g_1(\nu)$$

where $g_1(\nu) = \sup_{x \in \mathcal{X}_1} x^\top \nu - f_1(x)$ and $g_2(\nu) = \inf_{x \in \mathcal{X}_2} x^\top \nu - f_2(x)$. The domains Λ_1 and Λ_2 are defined as

$$\Lambda_1 = \{\nu \mid g_1(\nu) < \infty\}, \quad \Lambda_2 = \{\nu \mid g_2(\nu) > -\infty\}$$

In other words, the dual problem has a solution whose optimum is equal to the primal optimum.

Partial Proof

Start from the optimization problem

$$\begin{aligned} \inf_x \quad & f_1(x) - f_2(x) \\ & x \in \mathcal{X}_1 \cap \mathcal{X}_2 \end{aligned}$$

Convert this to a different but equivalent problem

$$\begin{aligned} \inf_{y,z} \quad & f_1(y) - f_2(z) \\ & y = z \\ & y \in \mathcal{X}_1 \\ & z \in \mathcal{X}_2 \end{aligned}$$

Introduce a dual variable to the constraint $y - z$ to arrive at the dual function

$$\begin{aligned} g(\nu) &= \inf_{\substack{y \in \mathcal{X}_1 \\ z \in \mathcal{X}_2}} f_1(y) - f_2(z) + \nu^\top (z - y) \\ & \inf_{z \in \mathcal{X}_2} (\nu^\top z - f_2(z)) - \sup_{y \in \mathcal{X}_1} (\nu^\top y - f_1(y)) \\ &= g_2(\nu) - g_1(\nu) \end{aligned}$$

Then the dual problem becomes

$$\sup_{\nu \in \Lambda_1 \cap \Lambda_2} g(\nu) = \sup_{\nu \in \Lambda_1 \cap \Lambda_2} g_2(\nu) - g_1(\nu)$$

To apply Fenchel duality to a conic optimization problem, set $f_1(x) = f(x)$ and $f_2(x) = 0$. Then $g_1(\nu) = \sup_{x \in \mathcal{X}}(x^\top \nu - f_1(x))$ and $g_2(\nu) = \inf_{x \in \mathcal{X}}(x^\top \nu)$. The second quantity can equivalently be expressed as

$$-g_2(\nu) = \sup_{x \in \mathcal{X}_2} (-x^\top \nu) = \begin{cases} 0 & \text{if } \nu \in \mathcal{X}_2^* \\ \infty & \text{otherwise} \end{cases}$$

where $\mathcal{X}_2^*$ denotes the dual cone of $\mathcal{X}_2$ defined by $\mathcal{X}_2^* = \{\nu \ : \ x^\top \nu \geq 0, \ \forall x \in \mathcal{X}_2\}$[22]. Thus, the dual problem becomes

[22] If $\nu \notin \mathcal{X}_2^*$ then $\exists \bar{x}$ such that $\bar{x}^\top \nu < 0$. Then $(-\alpha \bar{x})^\top \nu > 0$ if $\alpha > 0$. By making $\alpha \to \infty$ we have $(-\alpha \bar{x})^\top \nu \to \infty$.

$$\inf_{\nu} \quad g_1(\nu)$$
$$\nu \in \Lambda_1 \cap \mathcal{X}_2^*$$

where $-g_2(\nu) = 0$ because $\nu \in \mathcal{X}_2^*$. Thus, the dual variable belongs to the dual cone.

9 Robust Optimization

Consider a standard linear programming problem

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{s.t.} \quad Ax \le b$$

We typically assume that the data, given by c, A and b, are known. However, in practice, they are usually unknown. The main issue with this, is that LP problems are highly sensitive to the parameters. In effect, if we solve our nominal problem

$$\min_{x \in \mathbb{R}^n} \hat{c}^\top x \quad \text{s.t.} \quad \hat{A}x \le \hat{b}$$

Our nominal solution $\hat{x}^*$ may be infeasible if for example the row data $a_i = \hat{a}_i + \varepsilon|\hat{a}_i|$, for $\varepsilon \sim \mathcal{U}[-0.001, 0.001]$[23]. This is depicted in Figure 24, on which the nominal solution $\hat{x}^*$ is not feasible in the new domain.

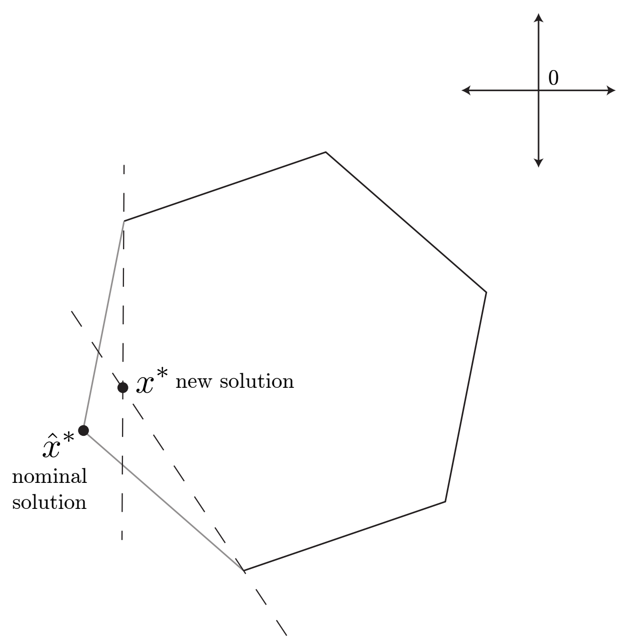

Figure 24: New domain formed by the unknown parameters. It can be observed that the nominal solution $\hat{x}^*$ is not even feasible in the new domain.

[23] This means that ε distributes uniform between $[-0.001, 0.001]$.

The alternative to deal with this issue is to consider the worst-case scenario.

9.1 Robust LP

Consider the nominal LP

$$\min_{x\in\mathbb{R}^n} \; c^\top x \quad \text{s.t.} \quad a_i^\top x \le b_i, \; i = 1, \dots, m$$

on which $(a_i, b_i) \in \mathcal{U}_i = \{\hat{a}_i + u \; : \; ||u||_2 \le \rho_i; \; b_i + v \; : \; ||v||_2 \le \mu_i\}$. The set defined by $\mathcal{U}_i$ is called **uncertainty set**.

We define the **Robust Counterpart** as

$$\begin{aligned} &\min_x \; c^\top x \\ &\text{s.t.} \;\; a_i^\top x \le b_i, \; \forall (a_i, b_i) \in \mathcal{U}_i, \; i = 1, \dots, m \end{aligned}$$

In other words, the robust counterpart requires the solution x^* to satisfy the tightest constraints possible under uncertainty. In the case of $c \in \mathcal{U}_0$ being unknown, we use an epigraphic reformulation to put it in the constraints

$$\begin{aligned} &\min_{x,t} \; t \\ &\text{s.t.} \;\; a_i^\top x \le b_i, \; \forall (a_i, b_i) \in \mathcal{U}_i, \; i = 1, \dots, m \\ &\qquad c^\top x \le t, \; \forall c \in \mathcal{U}_0 \end{aligned}$$

Reverting the epigraphic reformulation this can be written as

$$\begin{aligned} &\min_x \; \max_{c\in\mathcal{U}_0} \; c^\top x \\ &\text{s.t.} \;\; a_i^\top x \le b_i, \; \forall (a_i, b_i) \in \mathcal{U}_i, \; i = 1, \dots, m \end{aligned}$$

The above inequality constraint involving uncertainty in both a_i and b_i can be expressed more succinctly by first defining

$$z = \begin{bmatrix} a_i \\ b_i \end{bmatrix} \in \mathcal{Z}_i \quad \tilde{x} = \begin{bmatrix} x \\ -1 \end{bmatrix}$$

Then the "tightest" constraint can be re-expressed as

$$(\max_{z_i\in\mathcal{Z}_i} z_i^\top \tilde{x}) \le 0$$

Example 1: SVM
Consider the SVM problem with data $(x_i, y_i), \; i = 1, \dots, m$, on which $x_i \in \mathbb{R}^n$ and $y_i \in \{+1, -1\}$.

We want to construct a classifier of the form $\hat{y}(x) = \text{sign}(\omega^\top x)$. As an initial alternative we could try a least square problem of the form

$$\min_{w} \sum_{i=1}^{m} ||y_i - \text{sign}(\omega^\top x_i)||_2^2$$

However, this problem is not convex due to the sign function. Below is another loss function where correctly-classified samples yield zero loss while incorrectly classified samples yield loss that grows as the miss-classified sample is further away from the hyperplane.

$$\sum_{i=1}^{m} \Big(-y_i(\omega^\top x_i + b)\Big)_+$$

However, a total loss of zero can be obtained when $\omega^*, b^* = 0$. This problem can be addressed by introducing the following loss function

$$\min_{\omega \in \mathbb{R}^n} \sum_{i=1}^{m} (1 - y_i \omega^\top x_i)_+$$

on which the function $(\cdot)_+$ is defined as $(u)_+ = \max\{0, u\}$. This function is convex and is instead an upperbound on the original cost function. The term $(1 - y_i \cdot \omega^\top x_i)_+$, is in $[0, 1)$ if signs of y_i and $\omega^\top x$ coincide, and in $(1, \infty)$ if the signs do not.

Now, we may not know exactly the data, like a hyper-rectangle of the form $x_i : ||x_i - \hat{x}_i||_\infty \leq \rho_i$, we can write our Robust counterpart as:

$$\min_{\omega} \max_{\substack{||x_i - \hat{x}_i||_\infty \leq \rho_i \\ i=1,\ldots,m}} \sum_{i=1}^{m} (1 - y_i \omega^\top x_i)_+$$

This can be thought as a classic SVM problem, but on which each data point is depicted as a rectangle instead of a single point.

Observe that in a Robust LP, the term $a_i^\top x \leq b, \forall a_i \in \mathcal{U}_i$ can be properly replaced by

$$\max_{a_i \in \mathcal{U}_i} a_i^\top x = \varphi(x) \leq b_i$$

where $\varphi(x)$ is a convex function if $\mathcal{U}_i$ is convex, since $\varphi(x)$ is the pointwise maximum of affine functions on x over a convex domain. However, a nice property of Robust LP, is that even if $\mathcal{U}_i$ is not convex, the following property holds

$$\max_{a_i \in \mathcal{U}_i} a_i^\top x = \max_{a_i \in \text{conv}(\mathcal{U}_i)} a_i^\top x \ \leftarrow \ \text{convex}$$

That is we don't require $\mathcal{U}_i$ to be convex, since it can be properly replaced by its convex hull, maintaining intact the robust counterpart[24].

Example 2:
Consider the Robust LP

$$\begin{aligned}\min_{x\in\mathbb{R}^n} \ & c^\top x \\ \text{s.t.} \quad & a_i^\top x \leq b_i \\ & a_i \in \mathcal{U}_i \\ & i = 1, \dots m\end{aligned}$$

with a box uncertainty $\mathcal{U}_i$

$$\begin{aligned}\mathcal{U}_i &= \{a_i \ : \ ||a_i - \hat{a}_i||_\infty \leq \rho_i\} \\ &= \{\hat{a}_i + \rho_i u \ : \ ||u||_\infty \leq 1\}\end{aligned}$$

Then

$$\begin{aligned}\varphi_i(x) &= \max_{a_i \in \mathcal{U}_i} a_i^\top x \\ &= \hat{a}_i^\top x + \rho_i \max_{||u||_\infty \leq 1} u^\top x \\ &= \hat{a}_i^\top x + \rho_i ||x||_1\end{aligned}$$

With that, the robust counterpart can be written as

$$\min_{x\in\mathbb{R}^n} \ c^\top x \quad \text{s.t.} \quad \hat{a}_i^\top x + \rho_i ||x||_1 \leq b_i, \ i = 1, \dots, m$$

That can be properly written as an LP

$$\begin{aligned}\min_{x\in\mathbb{R}^n} \ & c^\top x \\ \text{s.t.} \quad & \hat{a}_i^\top x + \rho_i \sum_{j=1}^n s_j \leq b_i, \ i = 1, \dots, m \\ & - s_i \leq x_i \leq s_i, \ i = 1, \dots, n\end{aligned}$$

Example 3: Scenario Uncertainty
Consider the Robust LP with an scenario uncertainty set

$$\mathcal{U}_i = \{a_i^{(1)}, \dots, a_i^{(k_i)}\}$$

[24]The idea comes that if a point is robust feasible in $\mathcal{U}_i$ it will be feasible on $\text{conv}(\mathcal{U}_i)$. This is depicted on example 1.2.6 (page 12) on `https://www2.isye.gatech.edu/~nemirovs/FullBookDec11.pdf`.

Note that $\varphi(x) = \max_{a_i \in \mathcal{U}_i} a_i^\top x = \max_{1 \le j \le k_i} a_i^{(j)\top} x$. Its robust counterpart can be written as

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.} & \max_{1 \le j \le k_i} a_i^{(j)\top} x \le b_i, \ i = 1, \ldots, m \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \min_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.} & a_i^{(j)\top} x \le b_i, \ \forall j = 1, \ldots, k_i \\ \forall i = 1, \ldots, m & \end{array}$$

Example 4:
Recall the robust counterpart of the SVM problem

$$\min_{\omega} \max_{\substack{||x_i - \hat{x}_i||_\infty \le \rho_i \\ i=1,\ldots,m}} \sum_{i=1}^m (1 - y_i \omega^\top x_i)_+$$

Note that the uncertainty for the dataset is not coupled between datapoints. The uncertainty set can be written as

$$\begin{aligned} \mathcal{U}_i &= \{x_i \ : \ ||x_i - \hat{x}_i||_\infty \le \rho_i\} \\ &= \{\hat{x}_i + \rho_i u \ : \ ||u||_\infty \le 1\} \end{aligned}$$

For each i we have

$$\begin{aligned} \varphi_i(x) &= \max_{x_i \in \mathcal{U}_i} \sum_{i=1}^m (1 - y_i \omega^\top x_i)_+ \\ &= \sum_{i=1}^m \left(1 + y_i \max_{x_i \in \mathcal{U}_i} (-\omega^\top x_i)\right)_+ \\ &= \sum_{i=1}^m \left(1 - y_i \omega^\top \hat{x}_i + \rho_i ||\omega||_1\right)_+ \end{aligned}$$

Note that in the second line, the order of two maximizations are exchanged, which is a valid operation. In the third line, $x_i^* = -\text{sgn}(\omega)$, where sgn is an element-wise operation. Thus, the robust counterpart can be written as

$$\min_{\omega} \sum_{i=1}^m \left(1 - y_i \omega^\top \hat{x}_i + \rho_i ||\omega||_1\right)_+$$

This is not an easy problem to solve. As an alternative, since $(u + v)_+ \le (u)_+ + (v)_+$[25], a simpler problem can be solved as an upper bound of the previous one. Consider $u = 1 - y_i \omega^\top x_i$ and $v = \rho_i ||\omega||_1 \ge 0$. The following relaxation provides an upper bound of the robust counterpart of the SVM problem

[25] To see this bound, note that if both u and v are positive, the inequality is binding. If at least one of u or v is negative, then clearly the right hand side is greater than the left hand side.

$$\min_{\omega} \sum_{i=1}^{m} \left(1 - y_i \omega^\top \hat{x}_i\right)_+ + \left(\sum_{i=1}^{m} \rho_i\right) ||\omega||_1$$

that is widely implemented in packages like scikit.

The general steps of formulating a robust optimization can be summarized as

1. Specify the nominal problem, which in general form is

$$\begin{aligned}&\min_{x \in \mathbb{R}^n} \; f_0(x)\\ &f_i(x) \leq 0, \; i = 1, \ldots, m\end{aligned}$$

   Note that no equality constraints are allowed and the problem is assumed to be convex so that the robust counterpart can "inherit" the convexity of the problem.

2. For problem parameters or coefficients that are uncertain, define an uncertainty set. For example, for the constraint $a_i^\top x \leq b_i$, the uncertainty of a_i can be modeled as a "box" $u_i = a_i - \hat{a}_i$, where $||u_i||_\infty \leq \varepsilon$.

3. Arrive at the robust counterpart problem so that when solved, the solution x^* satisfies even the tightest of constraints possible from uncertainly. For a robust LP with uncertainty set example in step 2), the robust counterpart would be

$$\begin{aligned}&\min_{x \in \mathbb{R}^n} \; c^\top x\\ &(\hat{a}_i + u_i)^\top x \leq b_i, \; i = 1, \ldots, m\\ &u_i : ||u_i||_\infty \leq \varepsilon, \; i = 1, \ldots, m\end{aligned}$$

   that can be expressed as

$$\begin{aligned}&\min_{x \in \mathbb{R}^n} \; c^\top x\\ &\hat{a}_i^\top x + \varepsilon ||x||_1 \leq b_i, \; i = 1, \ldots, m\end{aligned}$$

   Generally, all constraints $f_i(\cdot, u)$ are assumed convex for each $u \in \mathcal{U}$. In robust LP, $f_i(\cdot, u)$ are linear for each fixed u.

When the uncertainty set is defined using a norm, the robust counterpart is easily defined. Generally, for an uncertainty set involving some norm of the form $\mathcal{A} = \{a : ||a - \hat{a}|| \leq \varepsilon\}$, the robust counterpart of the constraint $a^\top x \leq 0$ turns into $\max_{a \in \mathcal{A}} a^\top x = \hat{a}^\top x + \varepsilon ||x||_* \leq 0$. Recall that $||x||_*$ denotes the dual norm. Thus, if the uncertainty set is defined in terms of the $l2$ norm, then the constraint turns into that of a SOCP.

9.2 Intersection of Uncertainty Sets

Recall the box uncertainty set $\mathcal{A}_i = \{a_i \; : \; ||a_i - \hat{a}_i||_\infty \leq \rho_i\}$. In scenarios when it is unlikely for all components of a_i to reach their upperbound, the uncertainty is considered too large. Thus, it may be reasonable to restrict the uncertainty set to the intersection of two uncertainty sets. For example

$$\mathcal{U} = \{a \ : \ ||a - \hat{a}||_{(1)} \leq \alpha; ||a - \hat{a}||_{(2)} \leq \varepsilon\}$$

Example 1
Revisit the SVM problem. Let $x_i \in \mathbb{R}^n$ represent data points such that *a priori*, we know at most $k << m$ number of them have incorrect labels, where $\hat{y}_i \in \{-1, +1\}$. This can occur for instance, when working with a dataset of brain images diagnosed by physicians. The upper bound on the number of incorrect labels can be expressed as

$$\sum_{i=1}^{m} |y_i - \hat{y}_i| = ||y_i - \hat{y}_i||_1 \leq 2k$$

Thus, there is uncertainty in the labels. In robust optimization, we wish to find ω^*, b^* such that the largest loss possible resulting from uncertainty in y's is minimized.

$$\min_{\omega,b} \max_{\substack{y_i \in \{-1,1\} \\ ||y_i - \hat{y}_i||_1 \leq 2k}} \sum_{i=1}^{m} (1 - y_i(\omega^\top x_i + b))_+$$

Recall that non convex sets, such as $y_i \in \{-1, 1\}$, can be replaced by its convex hull $y_i \in [-1, 1]$, maintaining intact the robust counterpart. Thus, the robust counterpart can be written as

$$\min_{\omega,b} \max_{\substack{||y_i||_\infty \leq 1 \\ ||y_i - \hat{y}_i||_1 \leq 2k}} \sum_{i=1}^{m} (1 - y_i(\omega^\top x_i + b))_+$$

that is a Robust Optimization problem with the intersection of two uncertainty sets.

In general, an optimization under intersection of two uncertainty sets is expressed as

$$p^* = \max_{a \in \mathcal{A}_1 \cap \mathcal{A}_2} a^\top x$$

We define the support function of $a^\top x, \ a \in \mathcal{A}$ as

$$\varphi(x) = \max_{a \in \mathcal{A}} a^\top x$$

We assume that the support functions given by

$$\varphi_1(x) = \max_{a \in \mathcal{A}_1} a^\top x \qquad \wedge \qquad \varphi_2(x) = \max_{a \in \mathcal{A}_2} a^\top x$$

are easy to compute (for example if both uncertainty sets are defined by norms, like in the previous example).

By adding two slack variables we can re-express our optimization problem as

$$\max_{a,a_1,a_2} \; a^\top x$$
$$a = a_1 \in \mathcal{A}_1$$
$$a = a_2 \in \mathcal{A}_2$$

Using the Lagrangian, the original problem can be written as

$$p^* = \max_{a,a_1\in\mathcal{A}_1,a_2\in\mathcal{A}_2} \min_{\nu_1,\nu_2} a^\top x + \nu_1^\top (a_1 - a) + \nu_2^\top (a_2 - a)$$

The minimization and maximization operations can be exchanged if Slater conditions on the original problem are satisfied. For that we require that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are convex sets with a non-empty intersection and that $\mathcal{A}_1 \cap \mathcal{A}_2$ has a non-empty relative interior. Then, since the objective is affine, and Slater is satisfied, we can flip min and max to obtain:

$$\begin{aligned}
p^* &= \min_{\nu_1,\nu_2} \max_{a,a_1\in\mathcal{A}_1,a_2\in\mathcal{A}_2} a^\top x + \nu_1^\top (a_1 - a) + \nu_2^\top (a_2 - a) \\
&= \min_{\nu_1,\nu_2} \max_{a,a_1\in\mathcal{A}_1,a_2\in\mathcal{A}_2} a^\top (x - \nu_1 - \nu_2) + a_1^\top \nu_1 + a_2^\top \nu_2 \\
&= \min_{\nu_1,\nu_2} \left[\max_a a^\top (x - \nu_1 - \nu_2) + \underbrace{\max_{a_1\in\mathcal{A}_1} a_1^\top \nu_1}_{\varphi_1(\nu_1)} + \underbrace{\max_{a_2\in\mathcal{A}_2} a_2^\top \nu_2}_{\varphi_2(\nu_2)} \right] \\
&= \min_{\nu_1,\nu_2} \begin{cases} \varphi_1(\nu_1) + \varphi_2(\nu_2) & \text{if } x - \nu_1 - \nu_2 = 0 \\ +\infty & \text{otherwise} \end{cases} \\
&= \min_{\substack{\nu_1,\nu_2 \\ x=\nu_1+\nu_2}} \varphi(\nu_1) + \varphi(\nu_2) \\
&= \min_{\nu_2} \varphi_1(x - \nu_2) + \varphi_2(\nu_2)
\end{aligned}$$

We are interested then in the following constraint:

$$\max_{a\in\mathcal{A}_1\cap\mathcal{A}_2} a^\top x = \min_\nu \varphi_1(x - \nu) + \varphi_2(\nu) \leq b$$

Note that if in the last constraint we find a feasible $\bar{\nu}$ such that $\varphi_1(x - \bar{\nu}) + \varphi_2(\bar{\nu}) \leq b$, then for sure its minimum ν^* will satisfy the constraint. This implies that if we have a problem defined by:

$$\begin{aligned}
&\min_x \; c^\top x \\
&\text{s.t. } a^\top (x) \leq b, \; \forall a \in \mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2 \\
&\qquad x \in \mathcal{X}
\end{aligned}$$

can be recast as

$$\begin{aligned}
&\min_x \; c^\top x \\
&\text{s.t. } \min_\nu \varphi_1(x - \nu) + \varphi_2(\nu) \leq b \\
&\qquad x \in \mathcal{X}
\end{aligned}$$

that can be written as

$$\begin{aligned}
\min_{x,\nu}\ & c^\top x \\
\text{s.t. } & \varphi_1(x-\nu)+\varphi_2(\nu)\le b \\
& x\in\mathcal{X}
\end{aligned}$$

Example 2
Consider the following uncertainty sets:

$$\begin{aligned}
\mathcal{A}_{1,i} &= \{a : ||a-\hat{a}_i||_\infty \le \varepsilon_i\} \\
\mathcal{A}_{2,i} &= \{a : ||a-\hat{a}_i||_1 \le \kappa_i\}
\end{aligned}$$

As we know, their support functions are

$$\begin{aligned}
\varphi_{1,i}(x) &= \hat{a}_i^\top x + \varepsilon_i ||x||_1 \\
\varphi_{2,i}(x) &= \hat{a}_i^\top x + \kappa_i ||x||_\infty
\end{aligned}$$

Then the problem given by

$$\begin{aligned}
\min_{x}\ & c^\top x \\
\text{s.t. } & a_i^\top(x)\le b_i,\ i=1,\ldots,m,\ \forall a_i\in\mathcal{A}_i=\mathcal{A}_{1,i}\cap\mathcal{A}_{2,i}
\end{aligned}$$

Then, its robust counterpart can be written as

$$\begin{aligned}
\min_{x,\nu}\ & c^\top x \\
\text{s.t. } & \underbrace{\hat{a}_i^\top(x-\nu_i)+\varepsilon_i||x-\nu_i||_1}_{\varphi_{1,i}(x-\nu_i)} + \underbrace{\hat{a}_i^\top \nu_i+\kappa_i||\nu_i||_\infty}_{\varphi_{2,i}(\nu)} \le b_i,\ i=1,\ldots,m
\end{aligned}$$

Other examples of uncertainty models:

- Polytopes in terms of vertices $a^{(i)}, i=1,\ldots,m$. We define the polytope as $\mathcal{A}=\text{conv}\{a^{(i)}\}_{i=1}^m$. Then

$$\varphi_{\mathcal{A}}(x)=\max_{a\in\mathcal{A}} a^\top x = \max_{1\le i\le m} a^{(i)\top} x$$

- Polytopes in terms of $\mathcal{A}=\{a : Fa\ge g\}$

$$\begin{aligned}
\varphi_{\mathcal{A}}(x) &= \max_{a\in\mathcal{A}} a^\top x \\
&= \max_{a} \min_{\lambda\geq 0} a^\top x + \lambda^\top (Fa - g) \\
&= \min_{\lambda\geq 0} \max_{a} a^\top (x + F^\top \lambda) - \lambda^\top g \\
&= \min_{\lambda\geq 0} \begin{cases} -\lambda^\top g & \text{if } F^\top \lambda + x = 0 \\ +\infty & \text{otherwise} \end{cases} \\
&= \min_{\substack{\lambda\geq 0 \\ F^\top \lambda + x = 0}} \quad -\lambda^\top g
\end{aligned}$$

- Conic forms such as $\mathcal{A} = \{a : a = \hat{a} + P\zeta,\ \zeta \in K\}$, where K is a proper cone. We assume that the relative interior of the uncertainty set is not empty. Thus:

$$\begin{aligned}
\varphi_{\mathcal{A}}(x) &= \max_{a\in\mathcal{A}} a^\top x \\
&= \max_{\zeta\in K} \hat{a}^\top x + (P\zeta)^\top x \\
&= \hat{a}^\top x + \max_{\zeta\in K} \zeta^\top P^\top x
\end{aligned}$$

Using conic duality with dual variable λ:

$$\begin{aligned}
\varphi_{\mathcal{A}}(x) &= \hat{a}^\top x + \max_{\zeta} \min_{\lambda\in K^*} \zeta^\top P^\top x + \zeta^\top \lambda \\
&= \hat{a}^\top x + \min_{\lambda\in K^*} \max_{\zeta} \zeta^\top (P^\top x + \lambda) \\
&= \begin{cases} \hat{a}^\top x & \text{if } P^\top x + \lambda = 0, \lambda \in K^* \\ \infty & \text{otherwise} \end{cases} \\
&= \min_{\substack{\lambda\in K^* \\ P^\top x + \lambda = 0}} \quad \hat{a}^\top x
\end{aligned}$$

9.3 Chance Programming

Chance programming is an optimization problem involving constraints of the form

$$a^\top x \leq b$$

where $a \in \mathbb{R}^n$ follows some distribution $\mathcal{D}(\hat{a}, \Sigma)$. Thus, the problem involves solving for x such that

$$\mathbf{Prob}(a^\top x \leq b) \geq \eta = 1 - \epsilon$$

where ϵ is sometimes referred to as the reliability level. In other words, the problem involves finding x such that the probability that $a^\top x \leq b$ is high. This probability is usually difficult to calculate except for a few select distributions, such as the multivariate Gaussian. If $a \sim \mathcal{N}(\hat{a}, \Sigma)$, then $a^\top x = \mathcal{N}(\hat{a}^\top x, x^\top \Sigma x)$[26]. Then it follows from definition of cumulative distribution function of the

[26]This comes from definition of the multivariate normal. By definition, a random vector $Y \in \mathbb{R}^n$ is multivariate normal if every function $a^\top Y$ of Y has the univariate normal distribution. The mean and variance of $a^\top Y$ are given by $\mathbb{E}[a^\top Y] = a^\top \mu$ and $\mathrm{Var}(a^\top Y) = a^\top \mathrm{Cov}(Y) a = a^\top \Sigma a$.

standard normal distribution $\Phi(x)$ that

$$\mathbf{Prob}(a^\top x \le b) = \Phi\left(\frac{b - \hat{a}^\top x}{\sqrt{x^\top \Sigma x}}\right)$$

Given that $a^\top x \le b$ has to be satisfied with probability η or greater

$$\begin{aligned}\eta \le \Phi\left(\frac{b - \hat{a}^\top x}{\sqrt{x^\top \Sigma x}}\right) &\Leftrightarrow \Phi^{-1}(\eta)||\Sigma^{1/2}x||_2 \le b - \hat{a}^\top x \\ &\Leftrightarrow \hat{a}^\top x + \Phi^{-1}(\eta)||\Sigma^{1/2}x||_2 \le b\end{aligned}$$

Stating the distribution a follows $\mathcal{N}(\hat{a}, \sigma)$ equivalently means that a lives in the ellipsoid $\mathcal{U} = \{a : (a - \hat{a})^\top \Sigma^{-1}(a - \hat{a}) \le \Phi^{-1}(\eta)^2\}$, and this yields the same robust counterpart.

$$\begin{aligned}\mathcal{U} &= \{a : (a - \hat{a})^\top \Sigma^{-1}(a - \hat{a}) \le \Phi^{-1}(\eta)^2\} \\ &= \{a : ||\Sigma^{-1/2}(a - \hat{a})||_2^2 \le \Phi^{-1}(\eta)^2\} \\ &= \{(a, u) : u = \Sigma^{-1/2}(a - \hat{a}),\ ||u||_2^2 \le \Phi^{-1}(\eta)^2\} \\ &= \{(a, u) : a = \hat{a} + \Sigma^{1/2}u,\ ||u||_2 \le \Phi^{-1}(\eta)\} \\ &= \{\hat{a} + \Phi^{-1}(\eta)\Sigma^{1/2}u :\ ||u||_2 \le 1\}\end{aligned}$$

that yields

$$\begin{aligned}\varphi_{\mathcal{U}}(x) &= \max_{a \in \mathcal{U}} a^\top x \\ &= \hat{a}^\top x + \max_{||u||_2 \le 1} (\Phi^{-1}(\eta)\Sigma^{1/2}u)^\top x \\ &= \hat{a}^\top x + \Phi^{-1}(\eta) \max_{||u||_2 \le 1} u^\top(\Sigma^{1/2}x) \\ &= \hat{a}^\top x + \Phi^{-1}(\eta)||\Sigma^{1/2}x||_2\end{aligned}$$

In general, if $a \sim \mathcal{D}(\mu, \Sigma)$, a particular distribution, we want to determine the following set

$$\mathcal{P} = \{a : \mathbf{Prob}(a^\top x \le b) \ge 1 - \varepsilon\}$$

This is a hard problem to solve, and some techniques to address this issue are:

1) Large deviation theory: attempts to find lower bounds on $\mathbf{Prob}(a^\top x \le b)$, such that they can be used to add constraints in the optimization problem.

2) Distributional robustness: in this setting, the exact distribution is unknown, but the family of distribution $\mathcal{P}$ is known, then ensuring that the probability of $a^\top x \le b$ being greater than $\eta = 1 - \epsilon$ even if a comes from the most disadvantageous distribution from $\mathcal{P}$ can be formulated as

$$\inf_{p \in \mathcal{P}} \mathbf{Prob}_p(a^\top x \le b) \ge \eta = 1 - \epsilon \ \Leftrightarrow\ \sup_{p \in \mathcal{P}} \mathbf{Prob}_p(a^\top x \ge b) \le \varepsilon = 1 - \eta$$

Let $f(x,a) = a^\top x - b$, an equivalent of the chance constraint is given by

$$p(x) = \sup_{p \in \mathcal{P}} \mathbf{Prob}_p(a^\top x - b \geq 0) = \sup_{p \in \mathcal{P}} \mathbf{Prob}_p(f(x,a) \geq 0) \leq \varepsilon = 1 - \eta$$

A convex bound can be found in the following form. Since $\mathbf{Prob}_p(f(x,a) \geq 0) = \mathbb{E}[\mathbb{1}(f(x,a) \geq 0)]$, we can find an upper bound for $\mathbf{Prob}_p(f(x,a) \geq 0)$ with a specific convex function $\psi(s)$ such that

$$\mathbb{E}[\mathbb{1}(f(x,a) \geq 0)] \leq \mathbb{E}[\psi(f(x,a) \geq 0)]$$

An example of this is $\psi(s) = \exp(s)$ because $\exp(s) \geq \mathbb{1}(s)$ for $s \geq 0$.

To generalize the previous bound, let $\psi : \mathbb{R} \to \mathbb{R}$ a nonnegative valued, nondecreasing, convex function satisfying the following property: $\psi(z) > \psi(0) \geq 1$ for any $z > 0$. $\psi(z)$ is referred as *generating function*[27]. Denote $\mathbb{1}_A$ the indicator function of a set A, i.e., $\mathbb{1}_A(z) = 1$ if $z \in A$, and 0 otherwise. From the property of $\psi(z)$, a newly introduced free variable $t > 0$, and any random variable Z (with density $g_Z(z)$), we have the following bound

$$\begin{aligned}
\mathbb{E}[\psi(t^{-1}Z)] &= \int_{-\infty}^{\infty} \psi(t^{-1}z) g_Z(t^{-1}z) dz \\
\psi(t^{-1}z) \geq 0 \;\; \rightarrow \;\; &\geq \int_{0}^{\infty} \psi(t^{-1}z) g_Z(t^{-1}z) dz \\
\psi(t^{-1}z) > 1 \text{ for } z \geq 0 \;\; \rightarrow \;\; &\geq \int_{0}^{\infty} 1 \cdot g_Z(t^{-1}z) dz \\
&= \mathbb{E}[\mathbb{1}_{[0,\infty]}(t^{-1}Z)] \\
&= \mathbf{Prob}_p\{t^{-1}Z \geq 0\} \\
&= \mathbf{Prob}_p\{Z \geq 0\}
\end{aligned}$$

By taking $Z = f(x,a)$, from the previous inequality we obtain:

$$\pi(x) = \mathbf{Prob}_p\{f(x,a) \geq 0\} \leq \mathbb{E}[\psi(t^{-1}f(x,a))]$$

Then, $\mathbb{E}[\psi(t^{-1}f(x,a))] \leq \varepsilon$ implies $\pi(x) \leq \varepsilon$, since $\mathbb{E}[\psi(t^{-1}f(x,a))] \geq \pi(x)$. Multiplying the inequality by t the we obtain

$$t\mathbb{E}[\psi(t^{-1}f(x,a))] \leq t\varepsilon$$

Define $\Psi(x,t) = t\mathbb{E}[\psi(t^{-1}f(x,a)]$, that is a weighted linear combination of perspective functions $t\psi(t^{-1}f(x,a))$. Then, the previous inequality can be written $\Psi(x,t) \leq t\varepsilon \Leftrightarrow \Psi(x,t) - t\varepsilon \leq 0$. Since t is a free parameter, we can minimize over t: $\min_{t \geq 0} \Psi(x,t) - t\varepsilon = \min_{t>0} t\mathbb{E}[\psi(t^{-1}f(x,a)] - t\varepsilon$, to yield a closer bound for $\mathbf{Prob}_p(f(x,a) \geq 0\} = \pi(x) \leq \varepsilon$. Thus, imposing the constraint:

$$\min_{t>0} \Psi(x,t) - t\varepsilon \leq 0$$

can be used as a less tighter constraint of the original chance constraint $\mathbf{Prob}_p(f(x,a) \geq 0\} \leq \varepsilon$.

[27] More details on `https://pdfs.semanticscholar.org/73fc/c81a0698b391decc0799ea9cb2ff34632e9a.pdf`

Example 1:
From the above discussion, choose $\psi(z) = [(z+1)_+]^2$, that is nonnegative valued, nondecreasing, and convex that satisfies $\psi(0) = 1$ and $\psi(z) > \psi(0)$ for all $z > 0$. Then, for any $t > 0$:

$$\Psi(x,t) = t\mathbb{E}[(t^{-1}f(x,a)+1)_+^2] \leq t\varepsilon$$

will yield the desired change constraint. Dropping the + subscript (and hence obtaining an upper bound), yields a bound that only depends on the two first moments. That is:

$$t\mathbb{E}[(t^{-1}f(x,a)+1)^2] \leq t\varepsilon = t(1-\eta)$$

with a little algebra of expanding the square

$$2\mathbb{E}[f(x,a)] + t^{-1}\mathbb{E}[f(x,a)^2] + t\eta \leq 0$$

Minimizing over t to obtain the tightest constraint yields $t^* = \eta^{-1/2}\mathbb{E}[f(x,a)^2]^{1/2}$. Replacing t^*, we can recast our chance constraint as:

$$\mathbb{E}[f(x,a)] + (\eta\mathbb{E}[f(x,a)^2])^{1/2} \leq 0$$

Recall that in the linear case, $\mathbb{E}[f(x,a)] = \hat{a}^\top x - b$ and $\mathrm{Var}[f(x,a)] = x^\top \Sigma x = ||\Sigma^{1/2}x||_2^2$, and so $\mathbb{E}[f(x,a)^2] = \mathrm{Var}[f(x,a)] + (\mathbb{E}[f(x,a)])^2 = ||\Sigma^{1/2}x||_2^2 + (\hat{a}^\top x - b)^2$, obtaining:

$$\hat{a}^\top x - b + \eta^{1/2}(x^\top(\Sigma + \hat{a}\hat{a}^\top)x - 2b\hat{a}^\top x + b^2)^{1/2} = \hat{a}^\top x - b + \eta^{1/2}(x^\top\tilde{\Sigma}x - 2b\hat{a}^\top x + b^2)^{1/2} \leq 0$$

where $\tilde{\Sigma} = \Sigma + \hat{a}\hat{a}^\top$. This can be properly cast as a SOCP constraint:

$$\hat{a}^\top x - b + \eta^{1/2}\left|\left|\begin{bmatrix} z \\ y \end{bmatrix}\right|\right|_2 \leq 0$$

with $z = \tilde{\Sigma}^{1/2}x - b\tilde{\Sigma}^{-1/2}\hat{a}$ and $y = b(1 - \hat{a}^\top\tilde{\Sigma}^{-1}\hat{a})^{1/2}$ since:[28]

$$\begin{aligned}
\left|\left|\begin{bmatrix} z \\ y \end{bmatrix}\right|\right|_2 &= \left(||z||_2^2 + y^2\right)^{1/2} \\
&= \left[(\tilde{\Sigma}^{1/2}x - b\tilde{\Sigma}^{-1/2}\hat{a})^\top(\tilde{\Sigma}^{1/2}x - b\tilde{\Sigma}^{-1/2}\hat{a}) + b^2(1 - \hat{a}^\top\tilde{\Sigma}^{-1}\hat{a})\right]^{1/2} \\
&= \left[x^\top\tilde{\Sigma}x - 2b\hat{a}^\top x + b^2\hat{a}^\top\tilde{\Sigma}^{-1}\hat{a} + b^2 - b^2\hat{a}^\top\tilde{\Sigma}^{-1}\hat{a}\right]^{1/2} \\
&= (x^\top\tilde{\Sigma}x - 2b\hat{a}^\top x + b^2)^{1/2}
\end{aligned}$$

Example 2:
Consider the generating function $\psi(z) = \exp z$. We seek to compute

[28] More details on `https://web.stanford.edu/class/ee364a/lectures/chance_constr.pdf`

$$\sup_{p\in\mathcal{P}} \mathbf{Prob}_p(a^\top x - b > 0)$$

We assume that a is zero mean, the support of a is in $[-1,1]^n$ (i.e. the values of each a_i is bounded between $[-1,1]$), and each a_i's are independent of each other.

Consider the case of $t = 1$, and we got

$$\begin{aligned}
\Psi(x,1) &= \mathbb{E}[\exp(a^\top x - b)] \\
&= e^{-b}\mathbb{E}[\exp(a^\top x)] \\
&= e^{-b}\mathbb{E}\left[\exp\left(\sum_{i=1}^n a_i x_i\right)\right] \\
&= e^{-b}\mathbb{E}\left[\prod_{i=1}^n \exp(a_i x_i)\right] \\
&\underbrace{=}_{\text{indep.}} e^{-b}\prod_{i=1}^n \mathbb{E}[\exp(a_i x_i)]
\end{aligned}$$

Before continuing we will show the *Hoeffding Lemma*. Let Z a zero mean random variable bounded in $[-1,1]$. Note that the exponential is a convex function, that is below the line formed between -1 and 1. With that, for any $z \in [-1,1]$, the line is defined as $\mathcal{L} := \left\{\frac{e^\lambda - e^{-\lambda}}{2}(z-1) + e^\lambda, z \in [-1,1]\right\}$. With that

$$e^{\lambda z} \le \frac{e^\lambda - e^{-\lambda}}{2}(z-1) + e^\lambda = \frac{e^\lambda - e^{-\lambda}}{2}z + \frac{e^\lambda + e^{-\lambda}}{2}$$

Taking expectation on both sides for the random variable Z, and using that Z is zero mean[29]:

$$\begin{aligned}
\mathbb{E}[e^{\lambda Z}] &\le \frac{e^\lambda - e^{-\lambda}}{2}\mathbb{E}[Z] + \frac{e^\lambda + e^{-\lambda}}{2} \\
&= \frac{e^\lambda + e^{-\lambda}}{2} \\
&= \cosh(\lambda) \\
\underbrace{\to}_{\text{Taylor S.}} &= \sum_{k=0}^\infty \frac{\lambda^{2k}}{(2k)!} \\
&= \sum_{k=0}^\infty \frac{(\lambda^2)^k}{2^k} \cdot \frac{2^k}{(2k)!} \\
&\le \sum_{k=0}^\infty \left(\frac{\lambda^2}{2}\right)^k \cdot \frac{1}{k!} = \exp\left(\frac{\lambda^2}{2}\right)
\end{aligned}$$

Thus, let $\lambda = x_i$:

[29]The proof that $\frac{2^k}{(2k)!} \le \frac{1}{k!}$ can be found on `https://math.stackexchange.com/questions/2238069/how-is-the-factorial-of-2k-2k-k-1-3-5-2k-1/2238074`.

$$
\begin{aligned}
\Psi(x,1) &= e^{-b}\prod_{i=1}^{n}\mathbb{E}[\exp(x_i a_i)]] \\
&\leq e^{-b}\prod_{i=1}^{n}\exp\left(\frac{x_i^2}{2}\right) \\
&= \exp\left(\sum_{i=1}^{n}\frac{x_i^2}{2} - b\right) \leq 1\cdot\varepsilon
\end{aligned}
$$

Taking log we obtain:

$$
\frac{1}{2}\sum_{i=1}^{n} x_i^2 - b \leq \log\varepsilon
$$

that is a convex quadratic constraint on x.

For the case of t free (instead of $t = 1$), and minimizing over t: $\min_{t>0}\Psi(x,t) - t\varepsilon$, to obtain the closest bound of $p(x)$, the constraint can be written as

$$
\sqrt{2\log(1/\varepsilon)}||x||_2 \leq b
$$

To show this, recall that $\mathbb{E}[\psi(t^{-1}f(x,a))] \leq \varepsilon$ implies:

$$
\begin{aligned}
& \mathbb{E}[\exp(t^{-1}f(x,a))] \leq \varepsilon \\
\rightarrow\quad & \log\mathbb{E}[\exp(t^{-1}f(x,a))] \leq \log\varepsilon \\
\rightarrow\quad & t\log\mathbb{E}[\exp(t^{-1}f(x,a))] \leq t\log\varepsilon
\end{aligned}
$$

Thus, using the *Hoeffding Lemma* (HL)

$$
\begin{aligned}
t\log\mathbb{E}[\exp(t^{-1}f(x,a))] &= t\log\mathbb{E}[\exp(t^{-1}(a^\top x - b))] \\
&= t\log\left\{e^{-bt^{-1}}\mathbb{E}[\exp(a^\top(t^{-1}x))]\right\} \\
&= t\log\left\{e^{-bt^{-1}}\prod_{i=1}^{n}\mathbb{E}[\exp(t^{-1}x_i a_i)]\right\} \\
\text{using HL with } \lambda = t^{-1}x_i \ \rightarrow\quad &\leq t\log\left\{e^{-bt^{-1}}\prod_{i=1}^{n}\exp\left(\frac{x_i^2}{2t^2}\right)\right\} \\
&= t\log\left\{\exp\left(\sum_{i=1}^{n}\frac{x_i^2}{2t^2} - bt^{-1}\right)\right\} \\
&= \sum_{i=1}^{n}\frac{x_i^2}{2t} - b
\end{aligned}
$$

Then we want to solve:

$$\min_{t>0} \sum_{i=1}^{n} \frac{x_i^2}{2t} - b - t \log \varepsilon$$

Taking the derivative and setting it to zero

$$\begin{aligned} 0 &= -t^{-2} \sum_{i=1}^{n} \frac{x_i^2}{2} - \log \varepsilon \\ &= -t^{-2} \left(\sum_{i=1}^{n} \frac{x_i^2}{2} \right) + \log(1/\varepsilon) \\ \rightarrow \ t^* &= \left(\frac{1}{2\log(1/\varepsilon)} \sum_{i=1}^{n} x_i^2 \right)^{1/2} = \frac{1}{\sqrt{2\log(1/\varepsilon)}} ||x||_2 \end{aligned}$$

Replacing t^* we obtain:

$$\begin{aligned} \frac{1}{2t^*} \sum_{i=1}^{n} x_i^2 - b + t^* \log(1/\varepsilon) &= \frac{1}{2}\sqrt{2\log(1/\varepsilon)} ||x||_2 - b + \frac{1}{\sqrt{2\log(1/\varepsilon)}} ||x||_2 \log \varepsilon \\ &= \sqrt{2\log(1/\varepsilon)} ||x||_2 - b \\ &\leq 0 \end{aligned}$$

that is an SOCP constraint.

9.4 Moment Problems

We now consider optimization problems with constraints of the form $a^\top x \leq b$, where a follows some distribution with known moments. The Markov inequality is a useful inequality when considering these types of problems.

$$P(X \geq a) \leq \frac{\mathbb{E}[X]}{a}, \ X \geq 0 \tag{23}$$

The proof for Markov inequality is

$$\mathbb{E}[X] = \int x(s)p(s)ds \geq \int_{s \in \mathcal{A}} x(s)p(s)ds \geq a\mathbf{Prob}(A)$$

where $\mathcal{A} = \{x \ : \ x \geq a\}$. Now consider the primal moment problem where the first moment is given as $q_1, \dots q_m$.

$$
\begin{aligned}
&\inf_{\pi(\cdot)} \int f_0(s)\pi(s)ds \\
&\text{s.t.} \int \pi(s)ds = 1 \\
&\quad \int f_i(s)\pi(s)dx = q_i \;\; i = 1, \dots m \\
&\quad \pi(s) \geq 0 \;\forall s
\end{aligned}
$$

with $f_0, f_1, \dots, f_m \;\; : \mathbb{R}^n \to \mathbb{R}$. This problem is infinite-dimensional in its decision variable $\pi(s)$ with finite constraints. Converting between a primal and dual problem exchanges the number of variables and constraints. The Lagrangian function of this problem is

$$
\mathcal{L}(\pi, \nu) = \int f_0(s)\pi(s)dx + \sum_{i=1}^{m+1} \nu_i \left[q_i + \int f_i(s)\pi(s)ds \right]
$$

where $\nu \in \mathbb{R}^{m+1}$ because $\int \pi(s)ds = 1$ is another instance of $\int f_{m+1}(s)\pi(s)dx = q_{m+1}$ with $q_{m+1} = 1$ and $f_{m+1}(s) = 1$. The dual function then becomes

$$
g(\nu) = \min_{\pi(\cdot) \geq 0} \nu^\top q + \int \pi(s) \left[f_0(s) - \sum_{i=1}^{m+1} \nu_i f_i(s) \right] ds
$$

which in terms leads to the dual problem

$$
d^* = \max_{\nu} g(\nu) = \max_{\nu} \nu^\top q \;\; : \;\; f_0(s) \geq \sum_{i=1}^{m+1} \nu_i f_i(s)
$$

which is a convex, semi-infinite LP problem, with infinite number of constraints and finite number of variables. Usually $p^* \geq d^*$, but strong duality is achieved when the solution $\pi(\cdot)$ is in the interior of the feasible set ($\pi(\cdot) > 0$). A distribution that satisfies this is the Gaussian distribution.

Example 1:
Assume that f_i's are quadratic of the form:

$$
f_i(s) = \begin{bmatrix} s \\ 1 \end{bmatrix}^\top F_i \begin{bmatrix} s \\ 1 \end{bmatrix}, \; i = 0, \dots, m+1
$$

with $F_i = F_i^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ for every i and F_{m+1} is zero everywhere except at the component $(n+1, n+1)$ on where is equal to 1. Then, the dual problem can be cast as a SDP:

$$
d^* = \max_{\nu} g(\nu) = \max_{\nu} \nu^\top q \;\; : \;\; F_0 \succeq \sum_{i=1}^{m+1} \nu_i F_i
$$

Now consider a similar problem

$$\inf_{a\sim(\hat{a},\Gamma)} \mathbf{Prob}(a^\top x \geq b)$$

where $\Gamma \in \mathbb{S}^n$ is the covariance of a and $\mathbb{E}[a] = \hat{a}$. We can re-express the objective function as

$$\mathbf{Prob}(a^\top x \geq b) = \mathbb{E}_\pi(\mathbb{1}_\mathcal{A}(s))$$

where

$$\mathbb{1}_\mathcal{A}(s) = \begin{cases} 1 & \text{if } s \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{A} = \{a \ : \ a^\top x \geq b\}$. Thus, our decision variable changes from a to $s \in \mathbb{R}^n$. Since Γ and $\hat{a}$ are given, the second moment C can be directly solved as

$$C = \Gamma + \hat{a}\hat{a}^\top$$

Now, the original problem of finding $a \sim (\hat{a}, \Gamma)$ to minimize $\mathbf{Prob}(a^\top x \geq b)$ can now be written in terms of s as

$$\begin{aligned} \inf_{\pi(s)} \quad & \int \mathbb{1}_\mathcal{A}\pi(s)ds \\ \text{s.t.} \quad & \int \pi(s)ds = 1 \\ & \int s\pi(s)dx = \hat{a} \\ & \int ss^\top\pi(s)dx = C \end{aligned}$$

The three constraints can be even more concisely written as

$$\int \begin{bmatrix} s \\ 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}^\top \pi(s)ds = \begin{bmatrix} C & \hat{a} \\ \hat{a}^\top & 1 \end{bmatrix} = Q$$

Following the same logic in constructing the dual problem as in the first moment problem discussed, we get

$$\begin{aligned} \max_{M\in\mathbb{S}^{n+1}} \quad & \mathrm{Tr}(MQ) \\ \text{s.t.} \quad & \mathbb{1}_\mathcal{A}(s) \geq \begin{bmatrix} s \\ 1 \end{bmatrix}^\top M \begin{bmatrix} s \\ 1 \end{bmatrix} \forall s \end{aligned}$$

Since $\mathbb{1}_\mathcal{A}(s) = 1$ if $s^\top x \geq b$ and 0 otherwise, the above constraint is equivalent to

$$\begin{bmatrix} s \\ 1 \end{bmatrix}^\top M \begin{bmatrix} s \\ 1 \end{bmatrix} \leq 1 \quad \forall s$$

$$\begin{bmatrix} s \\ 1 \end{bmatrix}^\top M \begin{bmatrix} s \\ 1 \end{bmatrix} \leq 0 \quad \forall s, s^\top x < b$$

The first constraint can be cast as a LMI: $M \preceq J$, where $J = \mathrm{diag}(\mathbf{0}_n, 1)$, while the second constraint must be treated with the $\mathcal{S}$-procedure (see Boyd book page 655). Using the $\mathcal{S}$-procedure we have

$$\forall s, s^\top x \leq b \ : \ \begin{bmatrix} s \\ 1 \end{bmatrix}^\top M \begin{bmatrix} s \\ 1 \end{bmatrix} \leq 0$$

if and only if there exists $\tau \geq 0$ such that

$$M \preceq \tau \begin{bmatrix} 0 & a/2 \\ a^\top/2 & b \end{bmatrix}$$

Then, the dual can be cast as a SDP

$$\begin{aligned} \max_{M \in \mathbb{S}^{n+1}, \tau \geq 0} \quad & \mathrm{Tr}(MQ) \\ \text{s.t.} \quad & M \preceq J, \quad M \preceq \tau \begin{bmatrix} 0 & a/2 \\ a^\top/2 & b \end{bmatrix} \end{aligned}$$

If the covariance matrix satisfies $\Gamma \succ 0$, then strong duality holds, and the SDP bound is exact. The SDP problem can be solved analytically (see El Ghaoui & Nilim, 2005):

$$p^* = \frac{||\Gamma^{1/2}x||_2^2}{(b - \hat{a}^\top x)_+^2 + ||\Gamma^{1/2}x||_2^2}$$

9.5 Value-at-Risk (VaR) optimization

Consider $x \in \mathbb{R}^n$ a vector of returns (typically unknown), and $w \in \mathbb{R}^n$ a portfolio weight vector. Then $w^\top x$ represents the total return. To address the problem of finding an optimal portfolio $w^\star$ we have different approaches

1. **Mean-Variance approach:**

   Assume $\mathbb{E}[x] = \hat{x}$ and $\mathbb{E}[(x - \hat{x})(x - \hat{x})^\top] = \Sigma$. Our portfolio problem can be written as

   $$\max_{w \in \mathcal{W}} \ \hat{x}^\top w - \lambda w^\top \Sigma w$$

   where $\mathcal{W} = \{w \geq 0, \ \mathbf{1}^\top w = B\}$ and $\lambda \geq 0$ (B represents the initial budget). The term $\lambda w^\top \Sigma w$ control the risk of our portfolio by controlling the importance of the portfolio's variance in the optimization problem.

2. **VaR approach:**

We are interested in controlling the probability of bad returns. This can be expressed that we want that the probability of losing more than γ dollars to be less than a probability ε:

$$\mathbb{P}\{x : w^\top x \leq -\gamma\} \leq \varepsilon$$

In this case $\omega^\top x$ is our return, for $\gamma \geq 0$ the term $-\gamma$ represents that we are losing γ dollars. Thus, the constraint $\omega^\top x \leq -\gamma$ represents losing more than γ dollars. The previous constraint can equivalently be written as

$$\mathbb{P}\{x : \gamma \leq -w^\top x\} \leq \varepsilon$$

The VaR problem is written as:

$$\gamma^* = \min_{\gamma} \gamma \ \ : \ \ \mathbf{Prob}_x\{x : (-\omega)^\top x \geq \gamma\} \leq \varepsilon$$

that can be complicated to calculate even for known distributions of x. However, we consider the distributional robustness counterpart, on which we do not known the distribution of x and we only have information of the first two moments:

$$\begin{aligned} &\sup_{x \sim (\hat{x}, \Sigma)} \ \mathbf{Prob}\{x : (-\omega)^\top x \geq \gamma\} \leq \varepsilon \\ \Leftrightarrow &\inf_{x \sim (\hat{x}, \Sigma)} \ \mathbf{Prob}\{x : (-\omega)^\top x \leq \gamma\} \geq 1 - \varepsilon \end{aligned}$$

which for $\Sigma \succeq 0$ is equivalent as

$$(-\omega)^\top \hat{x} + \sqrt{\frac{1-\varepsilon}{\varepsilon}} \sqrt{w^\top \Sigma w} \leq \gamma$$

Since we are trying to minimize γ, we know it will be pushed to its lower bound. Then, solving

$$\begin{aligned} &\min_{w \in \mathcal{W}} (-\omega)^\top \hat{x} + \sqrt{\frac{1-\varepsilon}{\varepsilon}} \sqrt{w^\top \Sigma w} \\ &= \min_{w \in \mathcal{W}} -\omega^\top \hat{x} + \sqrt{\frac{1-\varepsilon}{\varepsilon}} ||\Sigma^{1/2} w||_2 \end{aligned}$$

which is a convex problem that finds optimal $\gamma^\star$ for known $\hat{x}$ and Σ.

We are interested in finding the VaR in the case where $\hat{x}$ is uncertain and is between bounds of the form $\hat{x} \in [\underline{x}, \bar{x}]$, while Σ is not affected by the uncertainty. Then, let $\kappa = \sqrt{(1-\varepsilon)/\varepsilon}$, the robust counterpart can be written as:

$$\begin{aligned} &\min_{w \in \mathcal{W}} \ \max_{\hat{x} \in [\underline{x}, \bar{x}]} \ - w^\top \hat{x} + \kappa \sqrt{w^\top \Sigma w} \\ \Leftrightarrow &\min_{w \in \mathcal{W}} \ \kappa \sqrt{w^\top \Sigma w} + \sum_{i=1}^{n} \max_{x_i \in [\underline{x}_i, \bar{x}_i]} -w_i x_i \\ \Leftrightarrow &\min_{w \in \mathcal{W}} \ \kappa \sqrt{w^\top \Sigma w} + \sum_{i=1}^{n} \max(-w_i \underline{x}_i, \ - w_i \bar{x}_i) \end{aligned}$$

that is a convex problem that can be directly implemented in cvx.

Now, we are interested in studying when there is uncertainty on Σ, and there is no uncertainty on $\hat{x}$. For example, a typical uncertainty of $\Sigma \in \mathbb{S}^3$ is of the form

$$\mathcal{C} = \left\{ \Sigma \;:\; \Sigma = \begin{bmatrix} \circ & + & ? \\ + & \circ & - \\ ? & - & \circ \end{bmatrix} \succeq 0 \right\}$$

where $\circ$ represents that the value is known, $+$ represents that the value is nonnegative and $-$ represents that the value is nonpositive. We are interested in finding the Robust Counterpart for this uncertainty of the VaR problem. First, consider that w is known, then the maximization part can be cast as an SDP, since:

$$\begin{aligned} & \max_{\Sigma \in \mathcal{C}} \; w^\top \Sigma w \\ \Leftrightarrow \; & \max_{\Sigma \succeq 0} \mathrm{Tr}(\Sigma(ww^\top)) \;:\; \Sigma_{11} = \circ, \Sigma_{22} = \circ, \Sigma_{33} = \circ, \Sigma_{12} \geq 0, \Sigma_{13} \leq 0 \end{aligned}$$

that is an SDP problem, since the objective function is linear on Σ, and we only have linear equalities and inequality constraints, while Σ has to be in the positive semidefinite cone.

Now, consider the robust counterpart:

$$\min_{w \in \mathcal{W}} \; \max_{\Sigma \in \mathcal{C}} \; - w^\top \hat{x} + \kappa \sqrt{w^\top \Sigma w}$$

To solve this problem, recall the following Theorem

Theorem 9.5.1. Sion Minimax Theorem. Let $f(x, y)$ be a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ (on which $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$), that fulfills the following properties

- $f(\cdot, y)$ is convex on its first argument for fixed y.
- $f(x, \cdot)$ is concave on its second argument for fixed x.
- $\mathcal{X}$ is a convex compact set.
- $\mathcal{Y}$ is a convex set.

Then:

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} f(x, y)$$

Proof: See `https://projecteuclid.org/euclid.kmj/1138038812`

Now, let $f(w, \Sigma) = -w^\top \hat{x} + \kappa \sqrt{w^\top \Sigma w} = -w^\top \hat{x} + \kappa ||\Sigma^{1/2} w||_2$. For fixed w, we have that the function is concave since its hypograph is a convex set. Indeed

$$\begin{aligned} & ||\Sigma^{1/2} w||_2 \geq t, \; \Sigma \succeq 0 \\ \Leftrightarrow \; & w^\top \Sigma w \geq t^2, \; \Sigma \succeq 0, t \geq 0 \\ \Leftrightarrow \; & t^2 - w^\top \Sigma w \leq 0, \; \Sigma \succeq 0, \; t \geq 0 \end{aligned}$$

that is an intersection of convex constraints, and hence a convex set.

For fixed Σ, we have that the function is convex since its epigraph is a convex set. Indeed

$$-w^\top \hat{x} + ||\Sigma^{1/2} w||_2 \leq t$$

is an SOCP constraint, that is convex. Then the problem is equivalent by flipping min and max

$$\max_{\Sigma \in \mathcal{C}} \ \min_{w \in \mathcal{W}} \ - w^\top \hat{x} + \kappa \sqrt{w^\top \Sigma w}$$

that can be cast as an SDP. For more details see "Worst-Case Value-At-Risk and Robust Portfolio Optimization: A Conic Programming Approach", by L. El-Ghaoui, M. Oks and F. Oustry, *Operations Research*, vol. 51, no. 4, pp. 543-556.

9.6 Applications

9.6.1 Stability on Discrete Time Systems

Consider the autonomous discrete time-invariant system $x(t+1) = Ax(t)$. It is well know that stability can be ensured if all eigenvalues of A are inside the unit circle.

Consider a possible Lyapunov matrix $P \in \mathbb{S}^n_{++}$ and $\alpha \in (0,1)$. If any trajectory satisfies

$$x(t+1)^\top P x(t+1) \leq \alpha^2 x(t)^\top P x(t)$$

then the system is stable. The previous condition can be cast as:

$$(Ax)^\top P (Ax) \leq \alpha^2 x^\top P x \ \Leftrightarrow \ A^\top P A \preceq \alpha P^2$$

with $P \succ 0$. This defines two LMIs that can be cast as a SDP:

$$\begin{aligned} \max_{P \in \mathbb{S}^n, t \in \mathbb{R}} \quad & t \\ \text{s.t.} \quad & tI \preceq P \\ & \alpha^2 P \succeq A^\top P A \\ & \mathrm{Tr}(P) = 1 \end{aligned}$$

If $t > 0$, then $P \succeq 0$, and so the system is stable. If $t < 0$ the system is not stable.

• Now, as an extension, consider a feedback discrete-time system:

$$x(t+1) = Ax(t) + Bu(t), \quad u(t) = Kx(t) \ \Rightarrow \ x(t+1) = (A + BK)x(t)$$

From the same argument as before if a matrix $P \succ 0$ exists such that:

$$\alpha^2 P \succeq (A + BK)^\top P (A + BK)$$

Define the following change of variables $X = P^{-1}$ and $U = KX$. Then, multiplying from the left by X and from the right by X, the system can be cast as:

$$
\begin{aligned}
& \alpha^2 X X^{-1} X \succeq X(A+BK)^\top X^{-1}(A+BK)X \\
\Leftrightarrow\ & \alpha^2 X \succeq (AX+BU)^\top X^{-1}(AX+BU) \\
\Leftrightarrow\ & \alpha^2 X - (AX+BU)X^{-1}(AX+BU) \succeq 0 \\
\Leftrightarrow\ & \begin{pmatrix} \alpha^2 X & (AX+BU)^\top \\ AX+BU & X \end{pmatrix} \succeq 0
\end{aligned}
$$

from Schur complements. This is a LMI on X and U, that is convex and can be cast as a SDP as

$$
\begin{aligned}
\max_{X\in\mathbb{S}^n, t\in\mathbb{R}} \quad & t \\
\text{s.t.} \quad & tI \preceq X \\
& \begin{pmatrix} \alpha^2 X & (AX+BKX)^\top \\ AX+BKX & X \end{pmatrix} \succeq 0 \\
& \mathrm{Tr}(X) = 1
\end{aligned}
$$

• Now, consider a Robust extension of the form:

$$
x(t+1) = A(t)x(t) + B(t)u(t)
$$

on which $A(t)$ and $B(t)$ are obtained from a possible set of L different modes:

$$
[A(t), B(t)] \in \left[A^{(1)}, B^{(1)}; A^{(2)}, B^{(2)}; \ldots; A^{(L)}, B^{(L)}\right]
$$

The system is robust stable if the same Lyapunov matrix P is stable for every mode in the closed loop:

$$
\begin{pmatrix} \alpha^2 X & (A^{(i)}X + B^{(i)}U)^\top \\ A^{(i)}X + B^{(i)}U & X \end{pmatrix} \succeq 0, \quad \forall i = 1, \ldots, L
$$

• Consider the system $x(t+1) = \varphi(Ax(t))$, where $\varphi(v) = \max(0, v)$ is the ReLU function. Observe that if $y = \varphi(u)$, then it clearly satisfies that $y_i \leq y_i u_i$, since for $u_i \leq 0$, we have $0 \leq 0$, and for $u_i \geq 0$, we have $u_i^2 \leq u_i^2$. Thus, the condition of stability can be cast as:

$$
x(t+1)^\top P x(t+1) \leq \alpha^2 x(t)^\top P x(t) \quad \text{whenever} \quad x_i(t+1)^2 \leq x_i(t+1)[Ax(t)]_i
$$

The constraint can be dualized to obtain a sufficient condition that yields a LMI:

$$
x(t+1)^\top P x(t+1) \leq \alpha^2 x(t)^\top P x(t) + \sum_{i=1}^{n} \lambda_i \left[x_i^2(t+1) - x_i(t+1)[Ax(t)]_i\right]
$$

• A similar scheme than the previous one are the so called Lure Systems. Assume a SISO system of the form $x(t+1) = Ax(t) + b\varphi(c^\top x(t))$, where $y(t) = c^\top x(t)$. For example, the function $\varphi(w)$ can be the ReLU function or the saturation function. The particularity of these functions are that their slope is always less than 1. Define the system as the following:

$$x(t+1) = Ax(t) + bu(t), \quad y(t) = c^\top x(t), \quad u(t) = \varphi(y(t))$$

We are interested in the stability of this system, that is, finding a matrix $P \succ 0$ such that:

$$x(t+1)^\top P x(t+1) \leq \alpha^2 x(t)^\top P x(t)$$

Since the slope of $\varphi(y)$ is always less than 1, we have:

$$\text{slope} \leq 1 \to \frac{u}{y} \leq 1 \Rightarrow u^2 \leq uy, u \geq 0$$

that is, we are interested in the problem:

$$\begin{aligned} q_1(x,u) &\triangleq (Ax+bu)^\top P(Ax+bu) - \alpha^2 x^\top P x \leq 0 \\ \text{s.t. } q_2(x,u) &\triangleq u^2 - u(c^\top x) \leq 0 \end{aligned}$$

Dualizing $-q_2(x,u) \geq 0$ we can write the system as a LMI of the form:

$$\begin{pmatrix} x \\ u \end{pmatrix}^\top \left\{ \begin{bmatrix} A^\top \\ b^\top \end{bmatrix} P \, [A \;\; b] - \begin{bmatrix} \alpha^2 P & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & c/2 \\ c^\top/2 & -1 \end{bmatrix} \right\} \begin{pmatrix} x \\ u \end{pmatrix} \leq 0$$

or equivalently:

$$\begin{bmatrix} A^\top \\ b^\top \end{bmatrix} P \, [A \;\; b] - \begin{bmatrix} \alpha^2 P & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & c/2 \\ c^\top/2 & -1 \end{bmatrix} \preceq 0$$

that, in addition of $P \succ 0$ and $\lambda \geq 0$, defines a SDP. This is actually a sufficient condition for Lyapunov stability, due to the $\mathcal{S}$-procedure lemma, that will be explored in the next sections.

• In the same approach of the previous problem, consider the problem of computing the maximum eigenvalue of C, that is:

$$p^* = \max_{x^\top x \leq 1} \; x^\top C x$$

Then using the previous approach, define $q_0(x) \triangleq x^\top C x$ and $q_1(x) \triangleq x^\top x - 1 \leq 0$. The Lagrangian of this maximization problem can be cast as:

$$\mathcal{L}(x,\lambda) = x^\top C + \lambda(1 - x^\top x)$$

From strong duality (due to Slater condition since $x = 0$ is a feasible interior point), we have:

$$p^* = d^* = \min_{\lambda \geq 0} \max_{x} \; x^\top C x + \lambda(1 - x^\top x)$$

that yields: $\lambda_{\max}(C) = d^* = \min_{\lambda \geq 0} \; \lambda \; : \; C \preceq \lambda I$.

9.6.2 Robust Stability on Discrete Time Systems

Consider the following system $x_{t+1} = (A + \Delta_t)x_t$, where $||\Delta_t|| \leq 1$. We want to know if Δ_t can destabilize the system.

Consider the Lyapunov function $V(x) = x^\top Px = ||P^{1/2}x||_2^2$ with $P \succ 0$. Lyapunov stability is ensured if for $\alpha \in (0, 1)$:

$$V(x_{t+1}) \leq \alpha^2 V(x_t)$$

As we know, without Δ_t, the condition reduces to the LMI: $A^\top PA \preceq \alpha^2 P$, $P \succ 0$.

Now, considering Δ_t we require:

$$\begin{aligned} &\underset{||\Delta||\leq 1}{\forall \Delta} : \quad (A+\Delta)^\top P(A+\Delta) - \alpha^2 P \preceq 0 \\ \Rightarrow\ &\underset{||\Delta||\leq 1}{\forall x, \Delta} : \quad x^\top (A+\Delta)^\top P(A+\Delta)x - \alpha^2 x^\top Px \leq 0 \end{aligned}$$

Add $y = \Delta x$ for some $||\Delta|| \leq 1$ as a new variable. Now recall that $||\Delta|| = ||\sigma||_\infty$, the maximum singular value of Δ. This implies that:

$$||\Delta|| = \max_{y=\Delta x} \frac{||y||_2}{||x||_2} \leq 1 \ \Rightarrow\ ||y||_2^2 \leq ||x||_2^2$$

Actually, for fixed y and x that satisfies $||y||_2^2 \leq ||x||_2^2$, the matrix Δ can be computed as $\Delta = yx^\top/(x^\top x)$. Indeed, $\Delta x = yx^\top x/(x^\top x) = y$. With this, our problem rewrites as:

$$\begin{aligned} \forall x, y :\ & q_0(x,y) \triangleq (Ax+y)^\top P(Ax+y) - \alpha^2 x^\top Px \\ \text{s.t. }\ & q_1(x,y) \triangleq y^\top y - x^\top x \leq 0 \end{aligned}$$

Theorem 9.6.1. $\mathcal{S}$-procedure. Let $q_0(z), q_1(z)$ quadratic forms:

$$q_i(z) = \begin{bmatrix} z \\ 1 \end{bmatrix}^\top Q_i \begin{bmatrix} z \\ 1 \end{bmatrix}$$

Assume $\exists z_0$ such that $q_1(z_0) < 0$. Then $q_0(z) \leq 0$ whenever $q_1(z) \leq 0$ if and only if:

$$\exists \tau \geq 0 \ :\ q_0(z) \leq \tau q_1(z),\ \forall z \qquad (\star)$$

or equivalently

$$\max_z \min_{\tau \geq 0} q_0(z) - \tau q_1(z) \leq 0$$

on which flipping min and max implies $(\star)$.

Proof: See Appendix B.1 and B.2 of Boyd's Convex Optimization book (page 653).

In our problem $z = (x, y)$. Thus, from the $\mathcal{S}$-procedure the problem can be equivalently written as:

$$(Ax + y)^\top P(Ax + y) - \alpha^2 x^\top Px \leq \tau(y^\top y - x^\top x)$$

or equivalently, $\exists \tau \geq 0$ such that:

$$\begin{bmatrix} A^\top PA - \alpha^2 P + \tau I & A^\top P \\ PA & -\tau I \end{bmatrix} \preceq 0$$

that is a SDP problem. If this is true, our perturbed system will always be stable.

Now we study a more general problem, on which we have multiple quadratic constraints of the form:

$$\begin{aligned} \max_{x,y} \ & q_0(x,y) \triangleq (Ax + y)^\top P(Ax + y) - \alpha^2 x^\top Px \leq 0 \\ \text{s.t.} \quad & x_i^2 \leq y_i^2, \quad i = 1, \dots, m \end{aligned}$$

Observe that since we have multiple quadratic constraints, the $\mathcal{S}$-procedure cannot be used. However, we can use weak duality to obtain a sufficient condition. Since weak duality will give us an upper bound of $q_0(x, y)$. If this upper bound is still less than zero, then our original system is indeed stable using that Lyapunov matrix. However, if the upper bound is greater than zero, we cannot conclude anything.

Let τ_i the dual variables associated to the constraints, then from weak duality we have:

$$\min_{\tau \geq 0} \max_{x,y} \ q_0(x,y) + \sum_{i=1}^{m} \tau_i (y_i^2 - x_i^2)$$

Using an epigraphic reformulation we have:

$$\begin{aligned} \min_{\tau \geq 0, t} \ & t \\ \text{s.t.} \quad & t \geq \max_{x,y} q_0(x,y) + \sum_{i=1}^{m} \tau_i (y_i^2 - x_i^2) \end{aligned}$$

that is equivalent:

$$\begin{aligned} \min_{\tau \geq 0, t} \ & t \\ \text{s.t.} \quad & t \geq (Ax + y)^\top P(Ax + y) - \alpha^2 x^\top Px + y^\top Sy - x^\top Sx, \quad \forall x, y \end{aligned}$$

where $S = \text{diag}(\tau)$. With this the problem can be cast as a SDP:

$$\begin{aligned} \min_{\tau \geq 0, t} \ & t \\ \text{s.t.} \quad & tI \geq \begin{bmatrix} A^\top PA - \alpha^2 P - S & A^\top P \\ PA & S \end{bmatrix} \end{aligned}$$

9.6.3 Robust Least-Squares

In this problem we are interested in solving the following version of the least squares problem:

$$\min_{x} \max_{||\Delta|| \leq 1} \quad ||(A + L\Delta R)x - y||_2^2 \leq t$$

where t is a predefined threshold to have a good performance. The matrices L and R are chosen such that Δ only affect some blocks, i.e. we have uncertainty of the matrix A only in certain parts.

Define $p = \Delta q$ and $q = Rx$, then the problem is equivalent to:

$$\min_{x,p} \; ||Ax + Lp - y||_2^2 \leq t \quad \text{whenever} \quad p^\top p \leq q^\top q = x^\top R^\top Rx$$

We can use the $\mathcal{S}$-procedure to obtain the following problem:

$$\exists \tau \geq 0, \forall x, p: \quad ||Ax + Lp - y||_2^2 - t \leq \tau(p^\top p - x^\top R^\top Rx)$$

following the same algebraic manipulation than the previous sections this problem can be cast as an SDP.

9.6.4 Robust SDP

Consider the nominal SDP:

$$\begin{aligned} &\min_{x} \; c^\top x \\ &\text{s.t. } F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0 \end{aligned}$$

where each $F_i \in \mathbb{S}^n$. We want to focus on uncertainty on $F(x)$.

a) Assume first that uncertain parameters $\delta \in \mathbb{R}^L$ affect the F data in a linear fashion:

$$F(x, \delta) = G(x) + L(x)\Delta R + R^\top \Delta L(x)$$

where $G(x) = F(x)$ (i.e. the nominal LMI), and Δ is of the form:

$$\Delta = \text{diag}(\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L})$$

where r_i is the dimension associated with the uncertain parameter δ_i.

b) We now want:

$$G + L\Delta R^\top + R\Delta L^\top \succeq 0, \quad \forall \Delta = \text{diag}(\delta_1 I_1, \ldots, \delta_L I_L) \text{ with } ||\delta||_\infty \leq 1$$

that is equivalent:

$$\forall z \ : \ z^\top (G + L\Delta R^\top + R\Delta L^\top) z \geq 0$$

Define $p = \Delta q$ and $q = R^\top z$. Then the problem can be cast as:

$$q_0(z,p) \triangleq z^\top (Gz + 2Lp) \geq 0, \ \forall z, p \quad \text{whenever} \quad p_i^2 \leq q_i^2 = (R^\top z)_i^2, \ i = 1, \dots, L$$

That can be written as

$$\begin{aligned} &\min_{z,p} \ q_0(z,p) \geq 0 \\ &\text{s.t.} \ p_i^2 \leq (R^\top z)_i^2, \quad i = 1, \dots, L \end{aligned}$$

A sufficient condition to satisfy this condition is using weak duality:

$$\max_{\lambda \geq 0} \min_{z,p} \ q_0(z,p) + \sum_{i=1}^{m} \lambda_i \left(p_i^2 - (R^\top z)_i^2 \right)$$

Defining $S = \text{diag}(\lambda)$ the condition can be cast as a SDP using an epigraphic reformulation.

Example 1:

Consider the following problem:

$$||A + B\Delta C|| \leq 1, \quad \forall \Delta \ \text{ such that } ||\Delta|| \leq 1$$

where $||A||$ is the induced two-norm. Recall that:

$$||A|| \leq 1 \ \Leftrightarrow \ A^\top A \leq I \ \Leftrightarrow \ \begin{bmatrix} I & A \\ A^\top & I \end{bmatrix} \succeq 0$$

Now, defining $F(\Delta) = A + B\Delta C$ we have:

$$\begin{aligned} ||F(\Delta)|| \leq 1 \Rightarrow &\begin{bmatrix} I & A + B\Delta C \\ (A + B\Delta C)^\top & I \end{bmatrix} \succeq 0 \\ \Rightarrow &\underbrace{\begin{bmatrix} I & A \\ A^\top & I \end{bmatrix}}_{G} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{L} \Delta \underbrace{\begin{bmatrix} 0 & C \end{bmatrix}}_{R^\top} + \begin{bmatrix} 0 \\ C^\top \end{bmatrix} \Delta \begin{bmatrix} B^\top & 0 \end{bmatrix} \succeq 0 \end{aligned}$$

Then, using the aforementioned approach:

$$\forall z : \ z^\top G z + 2 z^\top L p \geq 0 \quad \text{whenever} \quad p^\top p \leq q^\top q = z^\top R R^\top z$$

for $p = \Delta q$ and $q = R^\top z$. Using the $\mathcal{S}$-procedure we got:

$$\exists \lambda \geq 0 : \ z^\top G z + 2 z^\top L p \geq \lambda (z^\top R R^\top z - p^\top p)$$

that is equivalent to:

$$\exists \lambda \geq 0: \begin{bmatrix} G - \lambda R R^\top & L \\ L^\top & \lambda I \end{bmatrix} \succeq 0$$

Example 2: Stability on continuous time linear systems

Consider the system $\dot{x} = Ax$. It is well known that if exists a Lyapunov function of the form $V(x) = x^\top P x$, with $P \succ 0$, that satisfies that its Lie derivative $\dot{V}(x(t)) < 0$, then the system is asymptotically stable on the origin. This condition yields to the following conditions:

$$0 > \dot{V}(x(t)) = x(t)^\top (A^\top P + PA) x(t) \quad \Rightarrow \quad A^\top P + PA \prec 0,\ P \succ 0$$

Now consider the dynamic robust style system $\dot{x}(t) = (A + B\Delta(t)C)x(t)$ with $||\Delta(t)|| \leq 1,\ \forall t$. This defines a time-varying system $\dot{x} = A(t)x$ with $A(t) = A + B\Delta(t)C$. A possibility to ensure stability of this system is finding a unique P such that:

$$\forall \Delta, ||\Delta|| \leq 1 \ : \ (A + B\Delta C)^\top P + P(A + B\Delta C) \prec 0,\ P \succ 0$$

Repeating the process we have the following conditions:

$$\begin{bmatrix} A^\top P + PA + \lambda C C^\top & BP \\ B^\top P & -\lambda I \end{bmatrix} \prec 0, \quad P \succ 0$$

Applying Schur complements to the first LMI yields the Riccati equation.

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